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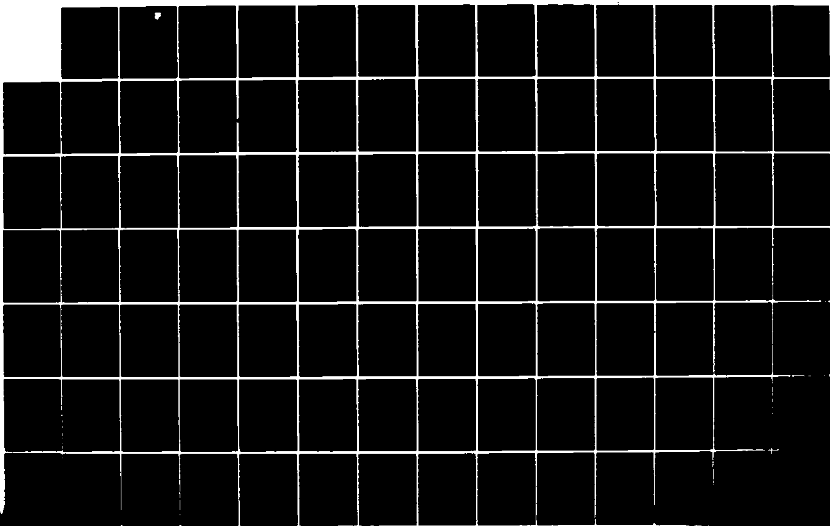

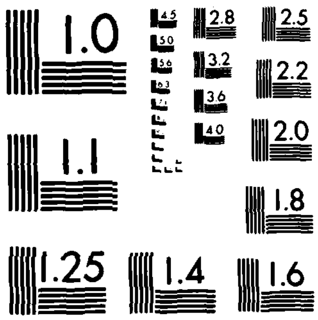

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RADC-TR-82-286, Vol IIIa (of six)
Final Technical Report
August 1984

BASIC EMC TECHNOLOGY ADVANCEMENT FOR C^3 SYSTEMS - Probabilistic Analysis of Combinational Circuits with Random Delays

AD-A150 349

Southeastern Center for Electrical Engineering Education

Abner Ephrath and Donald D. Weiner

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Random propagation delays are encountered in digital integrated circuits due to fluctuations in the fabrication process. These delays can be further increased due to the presence of electromagnetic interference. System performance can be evaluated from the expected values of the output signals. Analytical methods for determining the output expected values of combinational circuits with random delays are developed in this dissertation.

Given the input expectations, the network logic functions, and p.d.f.'s of the delays associated with the gates in the network, it is shown how to obtain the output expected values. Two types of delay elements are considered: (1) the pure delay elements, whose output is a delayed, but undistorted, replica of the input and (2) the discriminating delay element, where input rise and fall transitions experience different delays. Two degrees of network complexity are dealt with: (1) tree-like networks, in which there is only one path from every network input to any network output and (2) networks with reconvergent fanouts,

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where more than one path exists from some inputs to some outputs.

To simplify analysis of very large circuits, an approximate model is proposed where the circuits are subdivided into large logic blocks. The analytical techniques previously derived for individual gates are then applicable. Various strategies for characterizing the delays of the large logic blocks are considered and examined by means of computer simulations.

18. Subject Terms (Continued).

Fan Out Circuits
Reconvergent Fan Out
Tree-Like Networks
Electromagnetic Compatibility.

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GLOSSARY OF SYMBOLS

Notation

Upper-case is used to denote both the signal inputs and outputs of logic blocks (e.g., X3(t), Y2(t), Z2(t)). Upper-case x is reserved for the inputs to the first level of logic (e.g., X1(t), X2(t),...Xn(t)) while upper-case z is reserved for the outputs from the last level of logic (e.g., Z1(t), Z2(t),..,Zm(t)). Signals defined at points internal to a logic block are denoted by lower-case (e.g., x(t), y(t), z4(t)). When modeling logic blocks, delay elements are always placed at the output. The lower-case notation denoting the input to the delay element is always identical to the upper-case notation denoting the output (e.g., if the delay element input is denoted by z3(t) the delay element output will be denoted by Z3(t)).

A random variable (r.v.) is denoted by a symbol with an underbar (e.g., $\underline{\tau}$, $\underline{x1}$, $\underline{\alpha}$). A stochastic process is denoted as a time function with an underbar (e.g., $\underline{X2}(t)$, $\underline{y}(t)$, $\underline{Z}(t)$). The time variable is denoted by t. A specific time instant is denoted by t_k; k = 1,2,... Time delays are denoted by τ. Time derivatives are denoted by an upper dot (e.g., $\dot{y}(t)$, $\dot{X}1(t)$).

Probabilistic quantities are denoted as follows:

$F_{\underline{X}}(X) = \Pr\{\underline{X} < X\}$ = Cumulative Distribution function (c.d.f.)

$f_{\underline{X}}(X) = \frac{d}{dX} F_{\underline{X}}(X)$ = probability density function (p.d.f.)

$E[\underline{X}] = \int_{-\infty}^{\infty} X f_{\underline{X}}(X)dX$ = expected value of the r.v. $\underline{X}$.

$E[\underline{X}(t)]$ = expected value of the stochastic process $\underline{X}(t)$.

$\dot{E}[\underline{X}(t)] = \frac{d}{dt} E[\underline{X}(t)]$ = time derivative of the expected value of the process $\underline{X}(t)$.

$R_{\underline{XY}}(t_1,t_2) = E[\underline{X}(t_1)\underline{Y}(t_2)]$ = cross correlation function of the random variables $\underline{X}(t_1)$ and $\underline{Y}(t_2)$.

$R_{\underline{XY}}(\eta) = E[\underline{X}(t)\underline{Y}(t-\eta)]$ = cross correlation function when $\underline{X}(t)$ and $\underline{Y}(t)$ are mutually stationary random processes.

$\langle X(t)\rangle = \frac{1}{t_f-t_0}\int_{t_0}^{t_f} X(t)dt$ = time average of $X(t)$ over the finite observation time interval (t_o,t_f) .

$X(t)*Y(t) = \int_{-\infty}^{\infty} X(\theta)Y(t-\theta)d\theta$ = convolution operation between the functions $X(t)$ and $Y(t)$.

The notation associated with 0,1 binary signals is as follows (also, see Fig. 1.5):

u_i = time at which the $i^{\underline{th}}$ rise transition occurs

d_i = time at which the $i^{\underline{th}}$ fall transition occurs

$\upsilon_i = u_i-d_{i-1}$ = duration of the $i^{\underline{th}}$ gap

$\delta_i = d_i-u_i$ = duration of the $i^{\underline{th}}$ pulse

$U_i = u_i-u_{i-1}$ = duration of the $i^{\underline{th}}$ interval between successive rise transitions

$D_i = d_i-d_{i-1}$ = duration of $i^{\underline{th}}$ interval between two successive fall transitions.

$X^+(t)$ = counting signal for the rise transitions of $X(t)$

$X^-(t)$ = counting signal for the fall transitions of $X(t)$

$\underline{\tau}_{rT}$, $\underline{\tau}_{fT}$	Total delays for rise and fall output transitions in the longest path.
$\overline{\underline{\tau}}_r$, $\overline{\underline{\tau}}_f$	Mean path delays for rise and fall output transitions.
$\langle\underline{\tau}_r\rangle$, $\langle\underline{\tau}_f\rangle$	Weighted mean path delays for rise and fall output transitions.
$\underline{\tau}_{rG}$, $\underline{\tau}_{fG}$	Assigned Gaussian delays for rise and fall output transitions.
$E_i[\underline{Z}(t)]$	Output expected value calculated using i^{th} strategy for delay characterizations.
$Z_M(t)$	Reference waveform for $E[\underline{Z}(t)]$, obtained by Monte Carlo simulation.

The notation utilized in the discussion of switching functions is as follows:

X' = complement of switching variable X

$X \vee Y$ = OR operation between X and Y

$X \wedge Y = X \cdot Y$ = AND operation between X and Y

$(X \cdot Y)'$ = NAND operation between X and Y

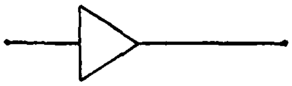
$(X \vee Y)'$ = NOR operation between X and Y

$X \oplus Y$ = XOR (exclusive OR) operation between X and Y

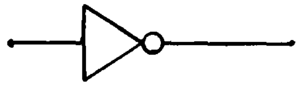
$X \odot Y$ = EXNOR (exclusive NOR) operation between X and Y

∇ denotes end of example

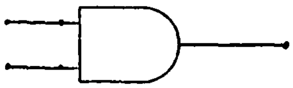
Schematic Symbols



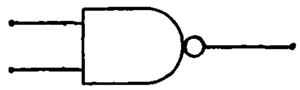
Buffer



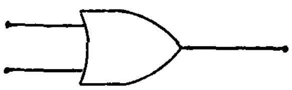
Inverter



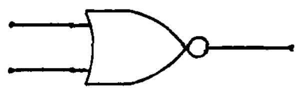
AND gate



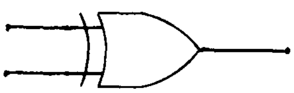
NAND gate



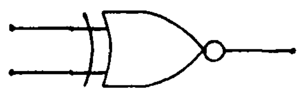
OR gate



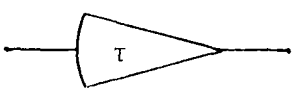
NOR gate



XOR gate



EXNOR gate



Pure delay

Discriminating delay

1. INTRODUCTION

1.1 Motivation

In recent years the trend has been towards the fabrication of microelectronic circuits capable of operating at increased speeds. By way of example, the goal of the VHSIC (Very High Speed Integrated Circuits) program is to achieve clock rates as high as 100 MHZ. As integrated circuits become faster, the occurrence of unintentional propagation delays within the chip can seriously affect system performance.

Because microelectronic circuits are highly complex, they are extremely difficult to analyze. The situation is further complicated when the circuits are exposed to electromagnetic interference (EMI). In an effort to simplify the analysis problem, a probabilistic approach has been proposed [1].[†] Using this point of view, the desired and interfering signals are treated as stochastic processes while the susceptibility level of the system is considered as a random variable. Approximations to probability density functions (p.d.f.'s) are obtained through measurements and statistical methods. These are then used to evaluate the probability of unacceptable system performance due to EMI.

The probabilistic nature of integrated circuits was observed in an experiment performed at Rome Air Development Center (RADC) [2]. The measured susceptibility levels of 252 "equivalent" 7400 TTL NAND gates were found to vary in a random fashion from 0.85 volts to a level in excess of 3.0 volts. Commonly used p.d.f's were shown to approximate the data.

[†] Numbers in brackets refer to listings in the list of references.

This dissertation is concerned with the increase in propagation delay caused by EMI, as observed by Alkalay and Weiner [3]. In their computer simulation sinusoidal EMI was injected into the output of a 7400 TTL NAND gate which was loaded by a fan out of 10 identical gates. The gate under test was driven by an input which caused the output to switch from the HIGH to LOW state in the absence of interference. As expected, for a large enough interfering signal, the gate output would not remain in the LOW state after switching. However, for smaller interfering signals, even though the gate output would remain in the LOW state after switching, increases in the propagation delay were noted. The increment in the delay was referred to as "excess propagation delay." A plot of excess propagation delay versus frequency is shown in Fig. 1.1 with the peak amplitude of the interfering signal as a parameter. Since typical propagation delays in the absence of interference are in the order of 10 nanoseconds. some of the observed excess propagation delays were considerably larger than the typical delays without interference.

An illustration of how unintentional delays can upset system performance is given in Example 1.1.

Example 1.1 Consider the simple combinational logic circuit shown in Figure 1.2. Assume a HIGH level is assigned the value 1 while a LOW level is assigned the value 0. Input B is assumed to be LOW for all time. The inputs at terminals A and C are denoted by X1(t) and X2(t), respectively. For the waveforms shown, the output Z(t) is identically 0.

In the presence of EMI, assume the output of gate 01 is delayed 1.5 ns while the output of gate 02 is delayed by 0.75 ns. The resulting waveforms

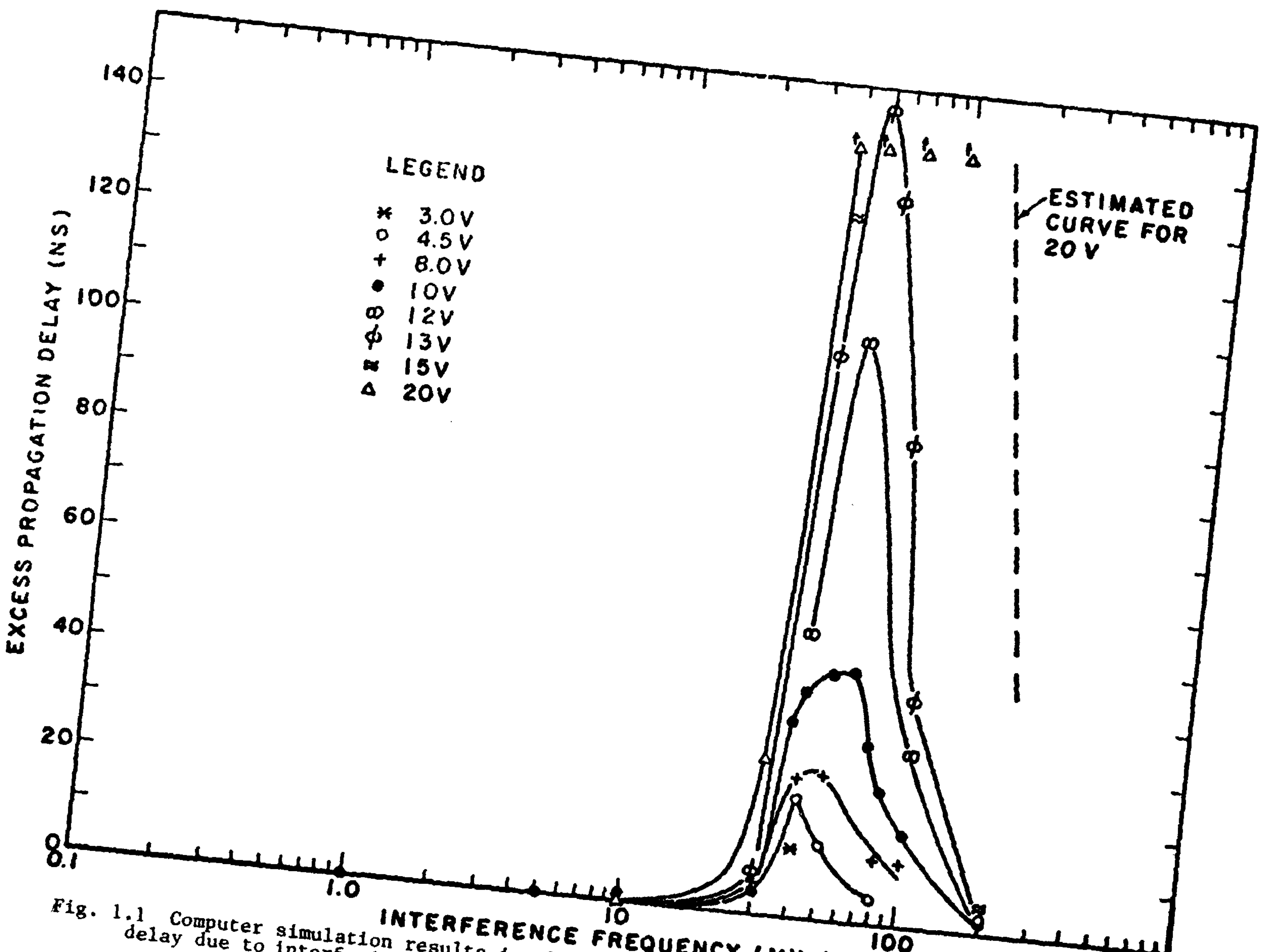


Fig. 1.1 Computer simulation results involving a 7400 TTL NAND gate. Sketch of excess propagation delay due to interfering signal with interference amplitude as a parameter [3].

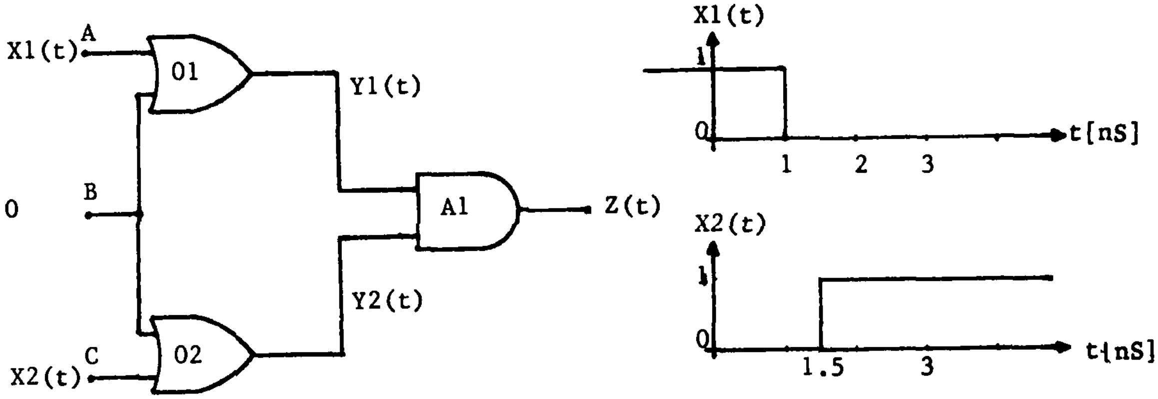


Fig. 1.2 Combinational logic circuit and signals for Example 1.1

and the output Z(t) are plotted in Figure 1.3. Because of the delays, a pulse is now present in Z(t). This is referred to as a hazard and could cause difficulty depending upon the use of the circuit. ∇

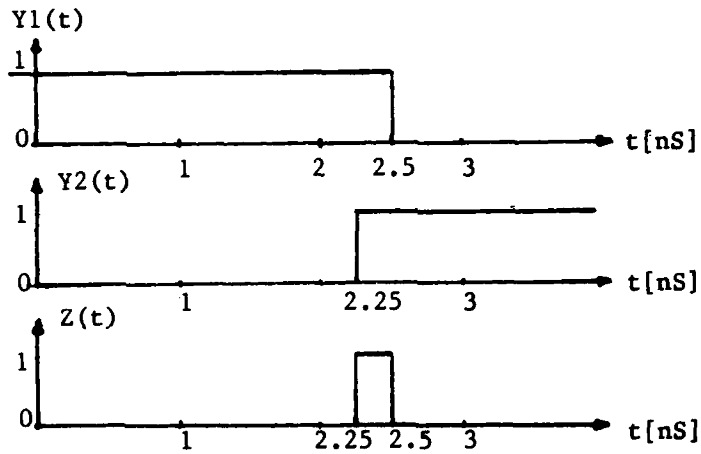


Fig. 1.3 Waveforms for the circuit of Figure 1.2 illustrating propagation delays due to the presence of EMI.

1.2 Objective

The objective of this work is to evaluate the performance of combinational logic circuits in which the signals experience random

propagation delays. Except for delays, the logic is assumed to perform in an ideal manner. Random propagation delays are present, even in the absence of EMI, due to parameter fluctuations in the fabrication process of integrated circuits. As pointed out previously, excess random propagation delays may also occur due to EMI. In this dissertation, both effects are combined together in a single overall random propagation delay.

If the delays associated with a combinational logic circuit are specified, it is treated as deterministic and a known input produces a known output. Denote the predicted output by $\hat{Z}(t)$. Specific values for the delays can be obtained in a variety of ways. For example, they may be measured, they may be chosen as the average of the maximum and minimum delays specified by the manufacturer, or, in a worst case design, chosen as the maximum delay quoted in the data sheets. In general, the delays are unknown and the actual output $\underline{Z}(t)$ may differ from the predicted output $\hat{Z}(t)$. One possible measure of system performance is the probability that $\underline{Z}(t)$ differs from $\hat{Z}(t)$.

For analytical simplicity, and without loss of generality, it is convenient to assign the value 1 to a HIGH level and the value 0 to a LOW level. Therefore, at any time instant, each waveform assumes either of the values 0 or 1. We refer to such waveforms as 0, 1 binary signals. Consider a combinational logic circuit in which the propagation delays have been specified. Assume $\hat{Z}(t_1) = 0$. The probability that $\underline{Z}(t_1)$ differs from $\hat{Z}(t_1)$ equals the probability that $\underline{Z}(t_1) = 1$. This is also the probability of error given that $\hat{Z}(t_1) = 0$.

We write

$$\Pr\{\text{error} \mid \hat{\underline{Z}}(t_1) = 0\} = \Pr\{\underline{Z}(t_1) = 1\} . \tag{1.1}$$

Now assume $\hat{Z}(t_2) = 1$. It follows that

$$\begin{aligned}\Pr\{\text{error} \mid \hat{\underline{Z}}(t_2) = 1\} &= \Pr\{\underline{Z}(t_2) = 0\} \\ &= 1 - \Pr\{\underline{Z}(t_2) = 1\}.\end{aligned} \tag{1.2}$$

In each case the conditional probability of error can be expressed in terms of the probability that the output equals unity. Note that the expected value of the output is given by

$$\begin{aligned}E[\underline{Z}(t)] &= 0 \cdot \Pr\{\underline{Z}(t) = 0\} + 1 \cdot \Pr\{\underline{Z}(t) = 1\} \\ &= \Pr\{\underline{Z}(t) = 1\}.\end{aligned} \tag{1.3}$$

As a result, the conditional probabilities of error can be expressed as

$$\begin{aligned}\Pr\{\text{error} \mid \hat{Z}(t_1) = 0\} &= E[\underline{Z}(t_1)] \\ \Pr\{\text{error} \mid \hat{Z}(t_2) = 1\} &= 1 - E[\underline{Z}(t_2)].\end{aligned} \tag{1.4}$$

Because of the above considerations, the effort in this dissertation is focused on evaluating the output expected value. Example 1.2 illustrates the application of equations (1.4).

Example 1.2 For a given input and a specified set of delays, assume the output $\hat{Z}(t)$ is derived deterministically to be the waveform shown in Figure 1.4(a). Taking the random nature of the propagation delays into

account, let the expected value of the output, $E[\underline{Z}(t)]$, be as depicted in Figure 1.4(b). The conditional probability of error given $\hat{Z}(t)$ is determined from Eq. (1.4) and is sketched in Figure 1.4(c). In general,

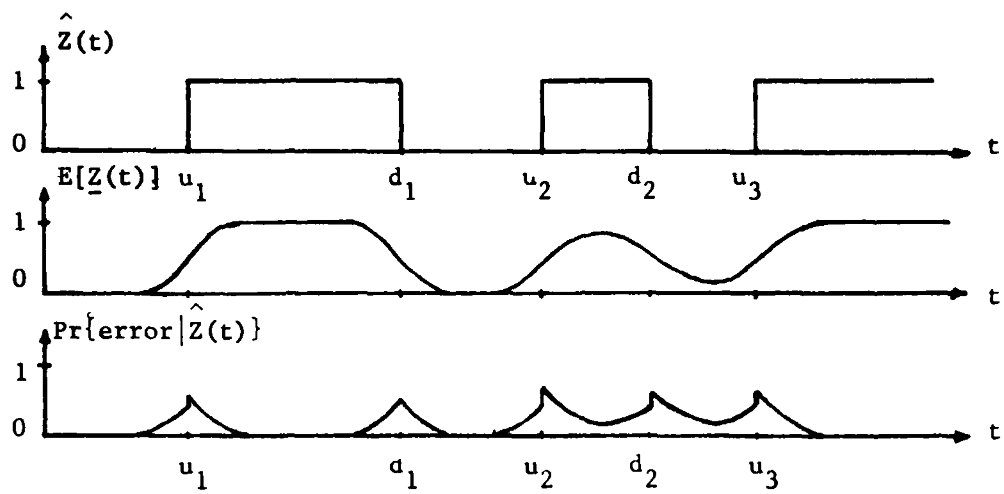


Fig. 1.4 Waveforms for Example 1.4.
(a) derived signal $\hat{Z}(t)$, (b) expected value $E[\underline{Z}(t)]$,
(c) conditional error Pr $\{error|\hat{Z}(t)\}$ as a function of time.

the conditional probability of error will be strongly dependent on the predicted output $\hat{Z}(t)$ which, of course, depends upon the specified delays and the given input. ∇

1.3 Problem Description

In this section the three basic considerations which enter into the problem description are discussed. They are: 1) the signals involved, 2) the subdivision of combinational logic circuits into logic blocks, and 3) the interconnection of the various logic blocks.

As mentioned in Sec. 1.2, all of the digital signals are assumed to be 0,1 binary (i.e., take on the values 0 and 1 with zero transition times

from one state to the other). A typical signal is shown in Fig. 1.5. The following notation is used:

u_i = time at which the i^{th} rise transition occurs.

(A rise transition is transition from 0 to 1.)

d_i = time at which the i^{th} fall transition occurs.

(A fall transition is a transition from 1 to 0.)

$\upsilon_i = u_i - d_{i-1}$ = duration of the i^{th} gap. (A gap corresponds to the interval between transitions during which the signal is 0).

$\delta_i = d_i - u_i$ = duration of the i^{th} pulse.

(A pulse corresponds to the interval between transitions during which the signal is 1).

$U_i = u_i - u_{i-1}$ = duration of i^{th} interval between two successive rise transitions.

$D_i = d_i - d_{i-1}$ = duration of i^{th} interval between two successive fall transitions.

By convention, the first fall transition occurring after t = 0 is

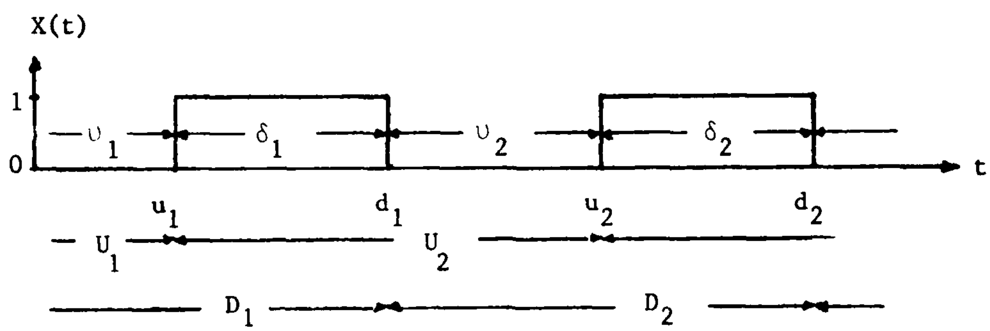


Fig. 1.5. Typical 0,1 binary signal

denoted by d_1. u_1 is the rise transition preceding d_1 and may or may not be positive. Both deterministic and stochastic 0,1 binary signals are included in this work. For deterministic signals the rise and fall transition times are assumed to be specified. On the other hand, u_i and d_i are viewed as random variables for stochastic processes.

Each 0,1 binary signal x(t) can be decomposed into two counting signals as illustrated in Fig. 1.6. A counting signal is a nondecreasing integer valued step-like waveform in which every step has unit height. $x^+(t)$ accounts for the rise transitions of x(t) while $x^-(t)$ accounts for the fall transitions. By definition,

$$x(t) = x^+(t) - x^-(t) . \tag{1.5}$$

Observe that a rise transition always precedes a fall transition, and vice versa. Symbolically,

$$\begin{aligned} u_i &< d_i < u_{i+1} \\ d_{i-1} &< u_i < d_i \end{aligned} \tag{1.6}$$

Only combinational logic circuits are considered in this work. The basic building blocks, which correspond to the smallest entities needed for the analysis, are gates, buffers and inverters. These are modeled as ideal logic circuits followed by ideal delay elements as shown in Fig. 1.7.

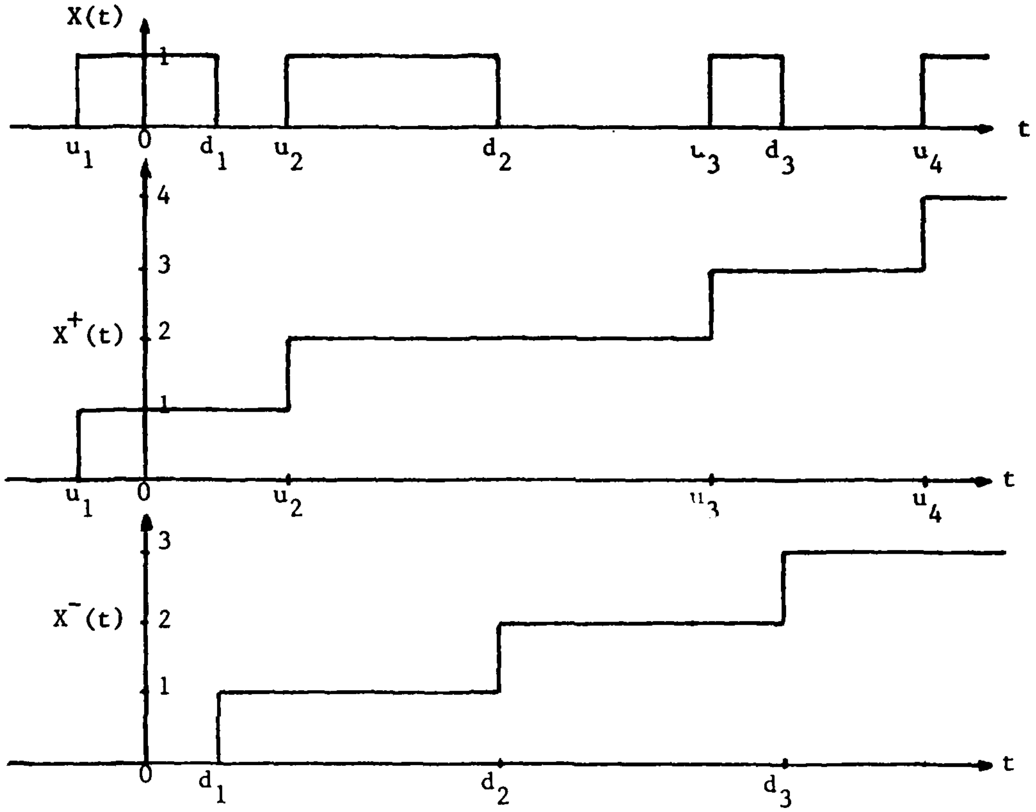


Fig. 1.6. Decomposition of a 0,1 binary signal into its two counting signals.

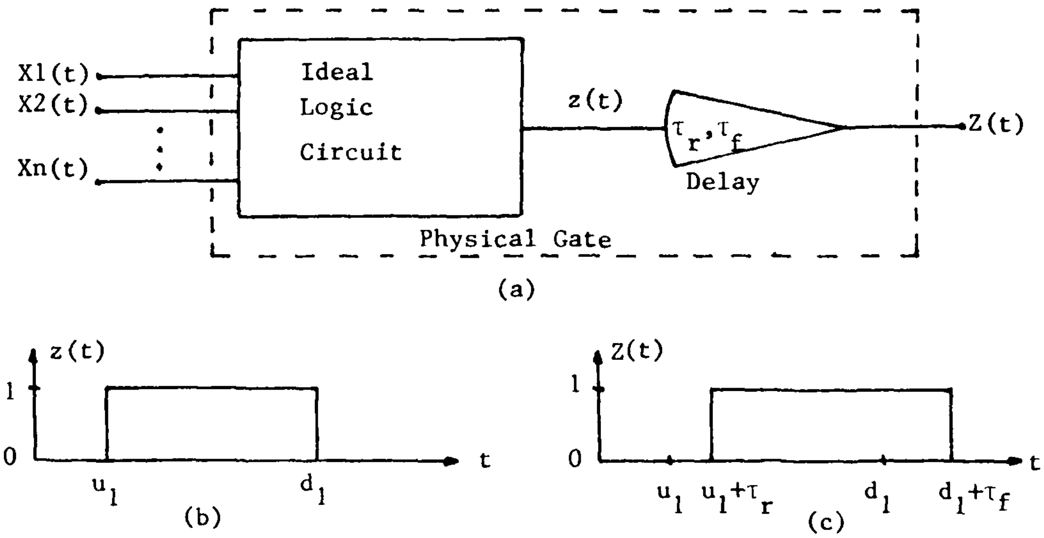


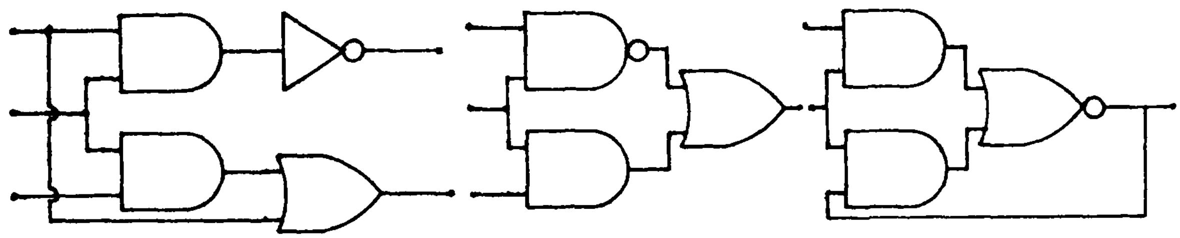
Fig. 1.7 a) Model of a physical gate, b) output from ideal logic circuit, c) output from discriminating delay circuit.

In practice, the delays τ_r experienced by rising transitions differ from the delays τ_f of the falling transitions (See Figs. 1.7(b) and (c)). Consequently, the ideal delay circuit is referred to as a discriminating delay circuit. The delays are assumed to be random variables whose randomness is caused by variation in the fabrication process plus the effects of EMI. The probability density function (p.d.f.'s) of $\underline{\tau}_r$ and $\underline{\tau}_f$ are assumed to be known.

For complicated combinational circuits it is convenient to subdivide the network into larger logic blocks. These are also modeled as ideal logic circuits followed by ideal delay elements. It is shown that the same analytical techniques used with the single logic gates can be extended to the more complicated logic blocks.

Various interconnections may arise in combinational circuits. The network is said to be tree-like when no more than one path exists from each input to every output (See Fig. 1.8(a)). The network is said to contain a reconvergent fanout when two or more branches, originating from a single point, merge together at some other point of the network. (See Fig. 1.8(b)). By definition, networks containing reconvergent fanouts are not tree-like. A feedback loop is said to exist in a network when the output of a logic block is fed back to a previous block whose output is connected in a path leading to an input of the original block. (See Fig. 1.8(c)). In this work only tree-like networks and networks containing reconvergent fanouts are considered.

Given a description of the input signals and the p.d.f's of the delays, the problem is to evaluate the expected value at the



(a) (b) (c)

Fig. 1.8 Examples of (a) tree-like network, (b) network with reconvergent fanout, (c) network with feedback.

network output. As pointed out previously, this can be used as a measure of the output error. Since the output may be used as the input to another combinational network, it is desirable that the analytical technique developed be capable of generating the input information required for analysis of the second circuit.

1.4 Problem Classification

A variety of problems may arise depending upon the signals, logic blocks, and networks. For the purpose of organization and clarity, a classification scheme is proposed in this section.

As mentioned previously, all signals are 0,1 binary. However, during a specified observation time they may be either

1) time-invariant (i.e., no transitions occur)

or

2) time variant (i.e., transitions do occur).

In addition, signals may be either

a) deterministic (i.e., all transition times are specified)

or

b) stochastic (i.e. transition times are considered as random variables with known probability density functions).

In this work the signals at the network inputs, referred to as primary inputs, are assumed to be deterministic. However, because deterministic signals become random after being processed by a logic block containing random delay, the analysis of logic blocks is performed assuming stochastic inputs.

In this work only zero memory logic blocks, such as gates, buffers, and inverters, are considered. They may be further classified according to whether the delays associated with rise and fall transitions are

1) equal (i.e., signals delayed without distortion)

or

2) unequal (i.e., signals delayed with distortion) .

and whether they are

a) zero (i.e. no delay)

b) specified (i.e. deterministic)

c) statistically independent random variables

d) statistically dependent for a given logic block but statistically independent from one logic block to another

e) statistically dependent .

With regard to the networks, only combinational networks are considered (i.e., no feedback loops exist). However, the networks may either have

1) no reconvergence (i.e., be tree-like)

or

2) one or more reconvergences.

In the ensuing discussion the proposed classification procedure will be used to characterize the various problems analyzed.

1.5 Literature Review

The earliest attempts at probabilistic analysis of digital logic circuits dealt with evaluation of the probability that a given switching expression (i.e., binary Boolean function) takes on the value 1. Fratta and Montanari [4] introduced algorithms for evaluating this probability using an orthogonal form of the switching expression. Parker and McCluskey [5,6] formalized the theory by presenting several theorems which show the relationship between Boolean operations and algebraic operations upon probabilities. Kumar and Breuer [7] derived additional results and showed how their approach is related to the Walsh coefficients and Reed-Muller canonic form of the switching expression. Murchland [8] considered multi-state systems and evaluated the probability of being in a state as well as the average transition rates between states. In particular, he showed how to use the Karnaugh map to determine the transition rates. Schneeweiss [9] presented a conceptually simple method which lends itself to efficient hand calculations for some problems. Debany [10] developed a computationally efficient computer program for evaluating the probability. The above investigators were motivated by a variety of applications. Fratta and Montanari were interested in computing the terminal reliability in a communication network. Parker and McClusky applied their results to the analysis of faulty logic circuits. Kumar and Breuer were interested in the random testing of digital circuits. Murchland was motivated by the reliability analysis of systems with numerous components. Schneeweiss

applied his results to the evaluation of fault trees used in reliability analysis while Debany was also interested in the random testing of digital networks. All of the work cited above is applicable to evaluation of the output expected value of a digital circuit without delay.

A first attempt at introducing delay was made by Coraluppi [11]. His approach is a graphical one in which deterministic pulses are mapped onto the complex plane by associating the time instants at which the pulses rise with the imaginary axis and the time instants at which the pulses fall with the real axis. A pulse, therefore, is represented by a vector in the complex plane. A discriminating delay, as described in Fig. 1.7, is accounted for by adding the vector $\tau_f + j\tau_r$ to the vector representing the pulse. Coraluppi gives an interesting graphical interpretation to the effects of delays associated with basic gates in the output of a digital system.

In recent years significant effort has been devoted to studying random delays in digital circuits by means of digital simulation. Magnhagen and Flisberg [12] characterize the delays by assigning Gaussian probability density functions (p.d.f.'s) to each delay. The means and variances are determined from the maximum and minimum delays specified by the manufacturer. Applications of their simulation program, called DIGSIM, to digital design verification of race conditions are given in [13]. It is important to recognize that DIGSIM does not perform Monte Carlo simulations. Instead probabilistic results are obtained by evaluating the mean and variance of the delay at the output and calculating desired probabilities from the resulting Gaussian p.d.f.

Another simulation program of interest is LDPS (LSI Delay Path Specification) which was recently developed by Al-Hussein and Dutton [14]. LDPS computes the propagation delays of both rise and fall transitions for all paths in an integrated circuit. It takes into account various factors which affect the delays such as loading, fabrication, and temperature. Empirical equations involving these factors were obtained using SPICE simulations. Other simulation programs are discussed by Grundmann [15] and Bass and Grundmann [16].

Major contributions to the probabilistic analysis of digital circuits containing random delays have been made by Grundmann [15], Bass and Grundmann [16], and Grundmann and Bass [17]. They discuss equal and unequal random propagation delays for rise and fall transitions in both combinational and sequential digital circuits. They have also written a computer program, called SLAP, which implements the theory. SLAP evaluates as a function of time the probability that the output of a digital circuit takes on the value 1 without the need to perform a Monte Carlo simulation.

The work in this dissertation expands on the excellent contributions made in [15] and [16]. First, many of the results in [15] and [16] are derived from a different point of view. The new derivations are more general and, it is believed, more easily understood. In addition, some of the approximations used by Grundmann and Bass when analyzing networks containing reconvergent fanouts have been removed by using higher order statistics. Finally, strategies are investigated for significantly reducing the amount of computation by subdividing a complex circuit into large blocks.

1.6 Dissertation Outline

Chapter 2 is devoted to the analysis of tree-like networks having statistically independent inputs. The case of pure delay (i.e., $\tau_r = \tau_f$) is discussed first. It is shown that $E[\underline{Z}]$, the expected value at the output of a logic block, is obtained by convolving $E[\underline{z}]$, the expected value of the output from the ideal logic circuit, with $f_{\underline{\tau}}(\tau)$, the p.d.f. of the pure delay element. The probability expression theorem is then introduced as a simple way for evaluating $E[\underline{z}]$ in terms of the expected values of the inputs.

The case of discriminating delay (i.e., $\tau_r \neq \tau_f$) is discussed next. After decomposing the 0,1 binary signals into the differences between their counting signals, an expression for $E[\underline{Z}]$ is derived which involves convolutions of the time derivatives $\dot{E}[\underline{z}^+]$ and $\dot{E}[\underline{z}^-]$ with $f_{\underline{\tau}r}(\tau_r)$ and $f_{\underline{\tau}f}(\tau_f)$, respectively. It is shown how to obtain arithmetic expressions for $\dot{\underline{z}}^+$ and $\dot{\underline{z}}^-$ in terms of time derivatives of the input signals by employing the arithmetic expression theorem.

Finally, a systematic method is developed for analyzing tree-like networks consisting of several logic blocks.

Chapter 3 discusses the analysis of combinational circuits containing reconvergent fanouts. The analysis is considerably more complicated because output higher-order moments are now required even though the primary inputs are statistically independent. As in Ch. 2, the pure delay and discriminating delay cases are discussed separately. For the pure delay case, the arithmetic expression theorem simplifies determination of the higher-order moments at the output of the ideal

logic circuit. For the case of discriminating delay, higher-order joint moments involving the time derivatives, $\dot{\underline{z}}^{+}$ and $\dot{\underline{z}}^{-}$, are needed. The same procedure used in Chapter 2 is employed for obtaining arithmetic expressions for $\dot{\underline{z}}^{+}$ and $\dot{\underline{z}}^{-}$. Having derived the basic approach, a method for analyzing combinational circuits with reconvergent fanouts is developed. Finally, an approximation utilized without discussion by a previous investigator [15] is analyzed and its limitations pointed out.

To simplify the analysis of very large circuits, it is proposed in Chapter 4 that the circuits be subdivided into large logic blocks containing several levels of logic. Guidelines for appropriate subdivision of the network are given. Possible approaches for placing the delay elements in the model of the large logic block are considered. The chosen approach, referred to as the simple output delay model, is justified in terms of simplicity and application of the model to situations involving EMI. Various strategies for characterizing the output random delays are presented and applied to a specific example. The results are compared with each other as well as with a computer generated Monte Carlo simulation.

Finally, a summary and suggestions for future work are given in Chapter 5.

2. ANALYSIS OF TREE-LIKE NETWORKS

2.1 Introduction

As mentioned in Chapter 1, only combinational logic circuits are considered in this work. When complicated combinational circuits are encountered, it will be convenient to subdivide the network into basic logic blocks. The notation to be used for the various signals is illustrated in Fig. 2.1. Following the practice of other investigators [14-17], basic logic blocks are modeled as ideal logic elements cascaded

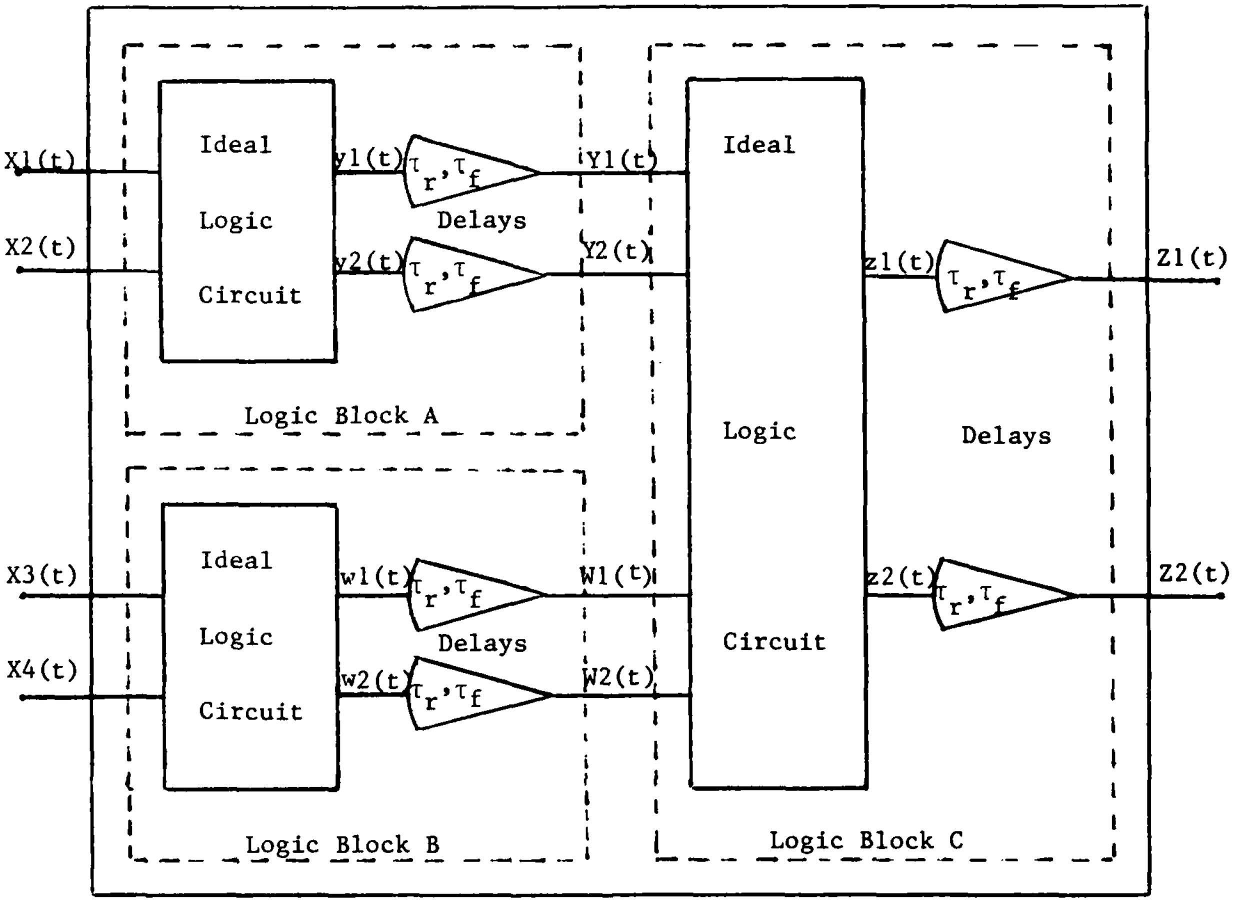


Fig. 2.1 A combinational circuit which has been subdivided into 3 basic logic blocks.

with ideal delay circuits. The ideal logic circuits are characterized by switching functions and the logic is performed without delay. The ideal delay circuit delays rise transitions by τ_r and fall transistions by τ_f. In general, $\tau_r \neq \tau_f$ and the ideal delay circuit is referred to as a discriminating delay. Observe that upper-case is used to denote both the signal inputs and outputs of logic blocks. In addition, upper case x is reserved for the inputs to the first level of logic while upper-case z is reserved for the outputs from the last level of logic. Signals defined at points internal to a logic block are denoted by lower-case. With respect to the discriminating delay elements, the lower case notation denoting their inputs is always identical to the upper-case notation denoting their outputs.

A typical input-output pair for a discriminating delay circuit is shown in Fig. 2.2. Note that the rise and fall transition instants of

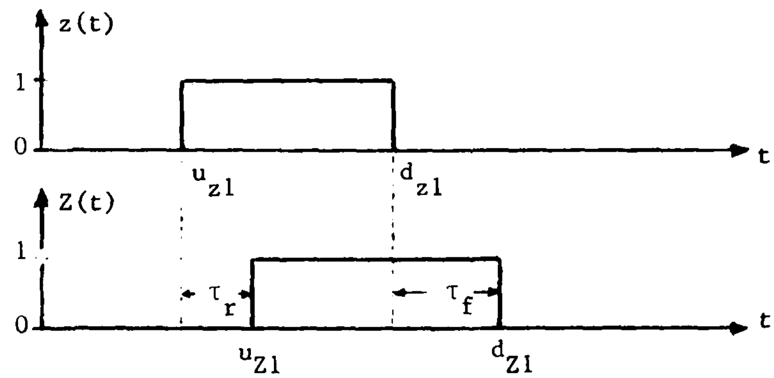


Fig. 2.2 A typical input-output signal pair for a discriminating delay circuit.

the output are related to those of the input by

$$u_{Zi} = u_{zi} + \tau_r \quad ; \ i = 1,2,...$$
$$d_{Zi} = d_{zi} + \tau_f \quad ; \ i = 1,2,... \tag{2.1}$$

A relatively simple case to analyze occurs when $\tau_r = \tau_f$. Then the delay circuit output is a delayed, but undistorted, version of the delay circuit input. This type of delay is referred to as a pure delay and is discussed first.

2.2 Pure Delay ($\tau_r = \tau_f$)

A typical logic block when $\tau_r = \tau_f = \tau$ is illustrated in Fig. 2.3. For ease of discussion, a single output $\underline{Z}(t)$ is considered. The input

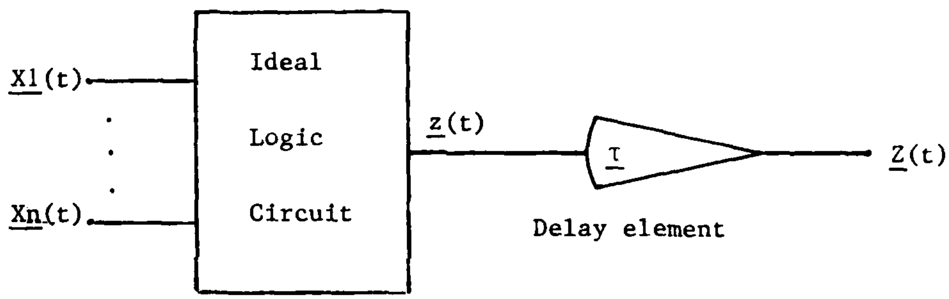


Fig. 2.3. A typical logic block with $\tau_r = \tau_f = \tau$.

signals $\underline{X1}(t),\ldots,\underline{Xn}(t)$ are assumed to be stochastic processes. Note that a stochastic process is denoted as a time function with an underbar. Similarly, a random variable (r.v.) is denoted as a symbol with an underbar. As discussed in section 1.2, system performance can be evaluated in terms of $E[\underline{Z}(t)]$.

Because $\underline{\tau}_r = \underline{\tau}_f = \underline{\tau}$, the output of the delay circuit is a delayed replica of its input. Specifically,

$$\underline{Z}(t) = \underline{z}(t-\underline{\tau}). \tag{2.2}$$

It follows that

$$E[\underline{Z}(t)] = E[\underline{z}(t - \underline{\tau})]. \tag{2.3}$$

The randomness associated with $\underline{z}(t-\underline{\tau})$ is due to the randomness of 1) the input signals $\underline{X1}(t),\ldots,\underline{Xn}(t)$ and 2) the delay $\underline{\tau}$. Assume the probability density function (p.d.f.) of $\underline{\tau}$ is known to be $f_{\underline{\tau}}(\tau)$. A convenient way to evaluate the expectation in (2.3) is to employ the conditional expectation theorem [18, p.208]. Let $E_{\underline{a}}[\underline{Y}]$ denote the expected value of $\underline{Y}$ with respect to the r.v. $\underline{a}$. Eq.(2.3) can then be written as

$$E_{\underline{X1}(t)\ldots,\underline{Xn}(t),\underline{\tau}}[\underline{Z}(t)] = E_{\underline{X1}(t),\ldots,\underline{Xn}(t),\underline{\tau}}[\underline{z}(t-\underline{\tau})]$$

$$= E_{\underline{\tau}}[E_{\underline{X1}(t),\ldots,\underline{Xn}(t)}[\underline{z}(t-\underline{\tau})|\underline{\tau} = \tau]]$$

$$= \int_0^\infty E_{\underline{X1}(t),\ldots,\underline{Xn}(t)}[\underline{z}(t-\underline{\tau})|\underline{\tau} = \tau)]f_{\underline{\tau}}(\tau)d\tau$$

$$= \int_0^\infty E_{\underline{X1}(t),\ldots,\underline{Xn}(t)}[\underline{z}(t-\tau)]f_{\underline{\tau}}(\tau)\,d\tau. \tag{2.4}$$

The latter result is recognized as the convolution of $E_{\underline{X1},\ldots,\underline{Xn}(t)}[\underline{z}(t)]$ with $f_{\underline{\tau}}(\tau)$. If * denotes convolution, Eq.(2.4) becomes

$$E_{\underline{X1}(t),\ldots,\underline{Xn}(t),\underline{\tau}}[\underline{Z}(t)] = E_{\underline{X1}(t),\ldots,\underline{Xn}(t)}[\underline{z}(t)]*f_{\underline{\tau}}(t). \tag{2.5}$$

For simplicity, subscripts on the expectation operator will be dropped when the meaning of the expectations is clearly understood. In its simplified form, Eq.(2.5) is

$$E[\underline{Z}(t)] = E[\underline{z}(t)] * f_{\underline{T}}(t) \quad . \tag{2.6}$$

<u>Example 2.1</u> A simple graphical example is shown in Fig. 2.4. Observe that the random delay alters system performance as reflected by the fact that $E[\underline{Z}(t)]$ differs from $E[\underline{z}(t)]$.

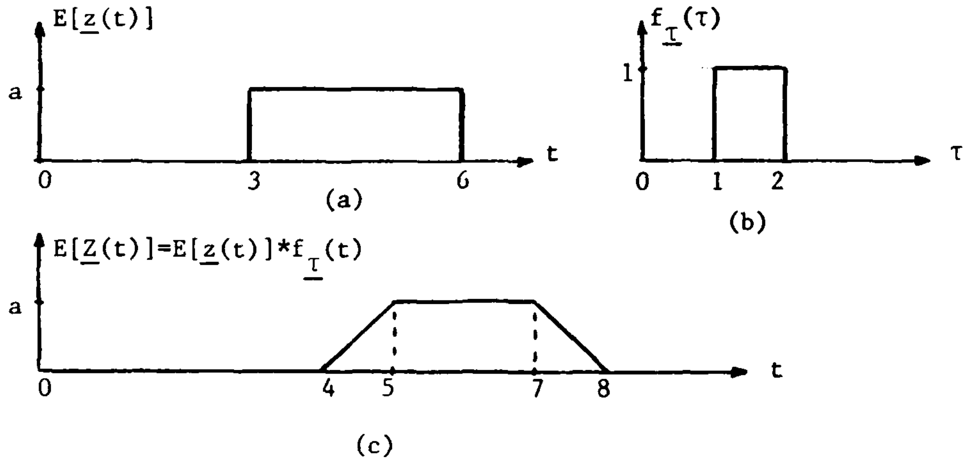


Fig. 2.4 Graphical example. (a) Expected value of input to delay circuit, (b) p.d.f. of random delay, (c) Expected value of output from delay circuit. ∇

A major step in evaluating $E[\underline{Z}(t)]$ according to Eq.(2.6) is the determination of $E[\underline{z}(t)]$. With reference to Fig. 2.3. the ideal logic circuit is characterized by its switching expression [19]-[21]

$$\underline{z}(t) = g(\underline{X1}(t),\ldots,\underline{Xn}(t)). \tag{2.7}$$

Three basic operations of switching algebra are:

NOT (Inversion, complement)
$$z = X' = \begin{cases} 0, & X = 1 \\ 1, & X = 0 \end{cases} \qquad (2.8)$$

AND
$$z = X1 \wedge X2 = X1 \cdot X2 = \begin{cases} 1, & X1=1 \text{ and } X2=1 \\ 0, & \text{otherwise} \end{cases} \qquad (2.9)$$

OR
$$z = X1 \vee X2 = \begin{cases} 0, & X1=0 \text{ and } X2=0 \\ 1, & \text{otherwise} \end{cases} \qquad (2.10)$$

It is well known that the NOT and AND operations form a complete set as do the NOT and OR operations. For convenience, all three operations are used in this work.

An implicant is defined to be a collection of switching variables or their complements connected by the AND operation (e.g., $X1 \wedge X2' \wedge X3$ is an implicant). Any switching function can be expressed as a set of implicants connected by the OR operation. This form is referred to as the "sum of products" form.

<u>Example 2.2</u> By definition, $z = g(X1, X2, X3, X4) = (X1' \vee X2 \cdot X4) \cdot X3'$ is not in a sum of products form. However, using the distributive law [19], the equivalent sum of products form is

$$z = X1' \cdot X3' \vee X2 \cdot X4 \cdot X3' \qquad \nabla \qquad (2.11)$$

Two implicants I_1 and I_2, are said to be nonoverlapping provided $I_1 \cdot I_2 = 0$ for all possible values chosen for the switching variables.

When the switching variables are random, it is necessary to evaluate $E[\underline{z}]$. Specifically, it is required to express $E[\underline{z}]$ in terms of the input expectations $E[\underline{X1}],\ldots,E[\underline{Xn}]$. This expression is referred to as the probability expression. Other investigators [5]-[7], [9], [15], [16] have demonstrated how this can be accomplished for specific situations. A general procedure is derived below.

Given the switching expression

$$\underline{z} = g_o(\underline{X1},\ldots,\underline{Xn}), \tag{2.12}$$

the first step in evaluating $E[\underline{z}]$ is to express Eq.(2.12) in a sum of products form. The second step is to modify the sum of products form into a sum of nonoverlapping implicants. This procedure is referred to as a Boolean orthogonalization [4] and is based on the relation

$$X \vee Y = X \vee X' \cdot Y \tag{2.13}$$

Since $X \cdot (X' \cdot Y) = 0$, the right side of Eq.(2.13) consists of nonoverlapping implicants whereas the left side does not. The Boolean orthogonalization procedure is carried out as follows:

1) Given $z = g_o(X1,\ldots,Xn)$ in a sum of products form, rearrange the product terms such that successive terms have an equal or larger number of variables. If several terms have the same number of variables, arrange them such that those having the smallest number of complemented variables come first. Represent the rearrangement by

$$g_o(X1,\ldots,Xn) = I_{10} \vee I_{20} \vee \ldots \vee I_{k0,0} \quad . \tag{2.14}$$

where k0 equals the number of implicants (product terms) in Eq.(2.14).

2) Define

$$z_1 = I_{10} \tag{2.15}$$

3) Construct

$$g_1(X_1,\dots,Xn) = I'_{10} \cdot g_o(X1,\dots,Xn)$$

$$= I'_{10} \cdot I_{10} \vee I'_{10} \cdot I_{20} \vee \dots \vee I'_{10} \cdot I_{k0,0}\,. \tag{2.16}$$

Use DeMorgan's law (i.e. $(X \cdot Y)' = X' \vee Y'$) and the distributive law (i.e. $(X \vee Y) \cdot Z = X \cdot Z \vee Y \cdot Z$) to expand the expression into a sum of products form. Where possible simplify the expression using the idempotency law (i.e., $X \vee X = X$ and $X \cdot X = X$) and the null law (i.e, $X \cdot X' = 0$). Finally, rearrange the terms according to step 1. Represent the rearrangement by

$$g_1(X1,\dots,Xn) = I_{11} \vee I_{21} \vee \dots \vee I_{k1,1} \tag{2.17}$$

where k1 equals the number of implicants in Eq.(2.17).

4) Define

$$z_2 = I_{10} \vee I_{11}. \tag{2.18}$$

5) Construct

$$g_2(X1,\dots,Xn) = I'_{11} \cdot g_1(X1,\dots,Xn)$$

$$= I_{11}' \cdot I_{11} \vee I'_{11} \cdot I_{21} \vee \dots \vee I'_{11} \cdot I_{k1,1}. \tag{2.19}$$

Simplify and rearrange $g_2(X1,\ldots,Xn)$ as in step 3 Represent the rearrangement by

$$g_2(X1,\ldots,Xn) = I_{12} \vee I_{22} \vee \ldots \vee I_{k2,2} \tag{2.20}$$

where k2 equals the number of implicants in Eq.(2.20).

6) Define

$$z_3 = I_{10} \vee I_{11} \vee I_{12}. \tag{2.21}$$

7) The procedure is repeated ℓ times until $g_\ell\ (X1,\ldots,Xn) = 0$.

The desired sum of nonoverlapping implicants is given by

$$z = g_o(x1,\ldots,Xn) = z_\ell = I_{10} \vee I_{11} \vee \ldots \vee I_{1(\ell-1)}, \tag{2.22}$$

as proven below.

Proof

The proof consists of showing that a) I_{10}, $I_{11}, \ldots$ $I_{1(\ell-1)}$ are nonoverlapping implicants and b) $z = z_\ell$.

(a) In general,

$$g_i\ (X1,\ldots,Xn) = I_{1i} \vee I_{2i} \vee \ldots \vee I_{ki,i}$$

$$= I'_{1(i-1)} \wedge g_{i-1}\ (X1,\ldots,Xn)$$

$$= I'_{1(i-1)} \wedge I'_{1(i-2)} \wedge \ldots \wedge I'_{10} \wedge g_o(X1,\ldots,Xn);$$

$$i=1,2,\ldots, \tag{2.23}$$

It follows that

$$g_i(X1,\ldots,X_n) \wedge I_{1k} = 0,\ k = 0,\ldots,(i-1). \tag{2.24}$$

Since I_{1i} is an implicant of g_i $(X1,\ldots,Xn)$,

$$I_{1i} \wedge I_{ik} = 0 \ ;\ k = 0,\ldots,\ (i-1) \quad . \tag{2.25}$$

Because Eq. (2.25) holds for all i, all the implicants of z_ℓ are nonoverlapping.

b) Mathematical induction is used to prove that $z = z_\ell$. With reference to equations (2.14) - (2.16), note that

$$z = g_o(X1,\ldots,Xn) = I_{10} \vee (I_{20} \vee \ldots \vee I_{k0,0})$$

$$= I_{10} \vee I'_{10} \cdot (I_{20} \vee \ldots \vee I_{k0,0})$$

$$= I_{10} \vee g_1(X1,\ldots,Xn) = z_1 \vee g_1(x1,\ldots,Xn). \tag{2.26}$$

Assume for j an integer between 2 and ℓ that

$$z = z_j \vee g_j\ (X1,\ldots,Xn). \tag{2.27}$$

Observe that

$$z = z_j \vee (I_{1j} \vee I_{2j} \quad \ldots \vee I_{kj,j})$$

$$= z_j \vee I_{1j} \vee I_{1j}{}' \cdot (I_{2j} \vee \ldots \vee I_{kj,j})$$

$$= z_{j+1} \vee g_{j+1}\ (X1,\ldots,Xn). \tag{2.28}$$

It follows that Eq.(2.28) holds for all j, $j=1,\ldots,\ell$. In particular,

$$z = z_\ell \vee g_\ell\,(X1,\ldots Xn). \qquad (2.29)$$

Since $g_\ell(X1,\ldots,Xn) = 0$, it is concluded that

$$z = z_\ell \qquad (2.30)$$

Example 2.3 Given the expression $z = g(X1, X2, X3, X4) = X1\cdot X2' \vee X3\cdot X4$, write the expression in a sum of nonoverlapping implicants form.

Step 1. The expression z is already given in a "sum of products" form and only rearrangement is required.

$$z = g_o(X1, X2, X3, X4) = X3 \cdot X4 \vee X1 \cdot X2' .$$

Step 2. $z_1 = X3 \cdot X4$.

Step 3. $g_1(X1, X2, X3, X4) = (X3 \cdot X4)'\cdot (X3 \cdot X4) \vee (X3 \cdot X4)'(X1 \cdot X2')$

$$= (X3 \cdot X4)'\cdot(X1 \cdot X2')$$

$$= (X3' \vee X4')\cdot(X1 \cdot X2')$$

$$= X3'\cdot X1 \cdot X2' \vee X4' \cdot X1 \cdot X2' .$$

Step 4. $z_2 = X3 \cdot X4 \vee X3'\cdot X1 \cdot X2'$

Step 5. $g_2(X1, X2, X3, X4) = (X3' \cdot X1 \cdot X2')'\cdot(X3'\cdot X1 \cdot X2')$

$$\vee (X3' \cdot X1 \cdot X2')' \cdot(X4' \cdot X1 \cdot X2')$$

$$= (X3'\cdot X1 \cdot X2')' \cdot(X4'\cdot X1 \cdot X2')$$

$$= (X3 \vee X1' \vee X2) \cdot (X4'\cdot X1 \cdot X2')$$

$$= X3 \cdot X4'\cdot X1 \cdot X2' .$$

Step 6. $z_3 = X3 \cdot X4 \vee X3' \cdot X1 \cdot X2' \vee X3 \cdot X4' \cdot X1 \cdot X2'$.

Step 7. $g_3(X1, X2, X3, X4) = (X3 \cdot X4' \cdot X1 \cdot X2')'(X3 \cdot X4' \cdot X1 \cdot X2') = 0$

Thus $z = z_3 = X3 \cdot X4 \vee X3' \cdot X1 \cdot X2' \vee X3 \cdot X4' \cdot X1 \cdot X2'$.

Note that the implicants are nonoverlapping. ∇

Having demonstrated how to express the switching expression as a sum of nonoverlapping implicants, the case in which the switching variables are random is now considered. Given

$$\underline{z} = g_o(\underline{X1},\ldots,\underline{Xn}) = \underline{z}_\ell = \underline{I}_{10} \vee \underline{I}_{11} \vee \ldots \vee I_{1(\ell-1)}, \tag{2.31}$$

it is necessary to evaluate $E[\underline{z}]$. Since the implicants in Eq.(2.31) are nonoverlapping, for any choice of values given the switching variables, only one implicant can assume the value 1. Therefore,

$$\begin{aligned} E[\underline{z}] &= P_r[\underline{z} = 1] = Pr[\underline{I}_{10} \vee \ldots \vee \underline{I}_{1(\ell-1)} = 1] \\ &= Pr[\underline{I}_{10}=1] + \ldots + Pr[\underline{I}_{1(\ell-1)} = 1] \\ &= E[\underline{I}_{10}] + \ldots + E[\underline{I}_{1(\ell-1)}]. \end{aligned} \tag{2.32}$$

The expected value of $\underline{z}$ is seen to equal the arithmetic sum of the expected values of the nonoverlapping implicants.

Each implicant of $\underline{z}_\ell$ is of the form

$$\underline{I}_{1k} = \underline{A}_{1k} \cdot \underline{A}_{2k} \cdot \ldots \cdot \underline{A}_{nk} \; ; \; k = 0,\ldots,(\ell-1). \tag{2.33}$$

where

$$\underline{A}_{jk} \in (\underline{X}_j, \underline{X}_j', 1) \; ; \; j = 1,\ldots,n. \tag{2.34}$$

Note that

$$E[\underline{I}_{1k}] = Pr[\underline{I}_{1k} = 1]$$

$$= Pr[\underline{A}_{1k} = 1, \underline{A}_{2k} = 1, \ldots, \underline{A}_{nk} = 1]. \tag{2.35}$$

Assuming the switching variables to be statistically independent, it follows that

$$E[\underline{I}_{1k}] = Pr[\underline{A}_{1k} = 1] \, Pr[A_{2k} = 1] \ldots Pr\,[\underline{A}_{nk} = 1]$$

$$= E[\underline{A}_{1k}] \, E[\underline{A}_{2k}] \ldots E[\underline{A}_{nk}]. \tag{2.36}$$

Substitution of Eq.(2.36) into Eq.(2.32) yields

$$E[\underline{z}] = E[\underline{A}_{10}] \, E[\underline{A}_{20}] \ldots E[\underline{A}_{n0}]$$

$$+ \ldots + E[\underline{A}_{1(\ell-1)}] \, E\,[\underline{A}_{2(\ell-1)}] \ldots E[\underline{A}_{n(\ell-1)}]. \tag{2.37}$$

The previous development suggests the following theorem for evaluation of the probability expression, $E[\underline{z}]$.

The Probability Expression Theorem

Let $\underline{z} = g_o(\underline{X1},\ldots,\underline{Xn})$ be a switching expression of the n statistically independent switching variables $\underline{X1},\ldots,\underline{Xn}$. If $\underline{z}$ is expressed as a sum of nonoverlapping implicants, then $E[\underline{z}]$ is readily obtained by interchanging each

1) OR operation, $\vee$, by the arithmetic operation of addition,
2) AND operation, $\cdot$, by the arithmetic operation of multiplication,
3) switching variable, $\underline{Xk}$, by its expected value.

Straightforward application of the probability expression theorem yields an expression for $E[\underline{z}]$ in terms of $E[\underline{Xk}]$ and/or $E[\underline{Xk}']$ where $k = 1,2,\ldots,n$. Expectations involving the complement of a switching variable can be eliminated by noting that

$$E[\underline{Xk}'] = Pr[\underline{Xk}' = 1] = Pr[\underline{Xk} = 0]$$

$$= 1 - Pr[\underline{Xk} = 1] = 1 - E[\underline{Xk}]. \qquad (2.38)$$

Example 2.4 Consider an ideal logic circuit with statistically independent inputs $\underline{X1}$, $\underline{X2}$, $\underline{X3}$, $\underline{X4}$. Given that the switching expression for the output is

$$\underline{z} = \underline{X1}' \vee \underline{X2} \cdot \underline{X3} \vee \underline{X4}, \qquad (2.39)$$

evaluate the probability expression, $E[\underline{z}]$.

Since the given switching expression is not a sum of nonoverlapping implicants, the first step is to perform the Boolean orthogonalization. Following the procedure outlined previously,

$$\underline{z} = \underline{X4} \vee \underline{X1}' \vee \underline{X2} \cdot \underline{X3}$$

$$\underline{z}_1 = \underline{X4},\ \underline{g}_1 = \underline{X4}' \cdot \underline{z} = \underline{X4}' \cdot \underline{X1}' \vee \underline{X4}' \cdot \underline{X2} \cdot \underline{X3}$$

$$\underline{z}_2 = \underline{X4} \vee \underline{X4}' \cdot \underline{X1}',\ \underline{g}_2 = (\underline{X4}' \cdot \underline{X1}')' \cdot (\underline{X4}' \cdot \underline{X2} \cdot \underline{X3})$$

$$= (\underline{X4} \vee \underline{X1}) \cdot (\underline{X4}' \cdot \underline{X2} \cdot \underline{X3}) = \underline{X1} \cdot \underline{X2} \cdot \underline{X3} \cdot \underline{X4}'$$

$$\underline{z}_3 = \underline{X4} \vee \underline{X1}' \cdot \underline{X4}' \vee \underline{X1} \cdot \underline{X2} \cdot \underline{X3} \cdot \underline{X4}',\ \underline{g}_3 = 0$$

$$\therefore\ \underline{z} = \underline{z}_3 = \underline{X4} \vee \underline{X1}' \cdot \underline{X4}' \vee \underline{X1} \cdot \underline{X2} \cdot \underline{X3} \cdot \underline{X4}' \qquad (2.40)$$

Direct application of the probability expression theorem to Eq.(2.40) results in

$$E[\underline{z}] = E[\underline{X4}] + E[\underline{X1}'] E [\underline{X4}'] + E[\underline{X1}] E [\underline{X2}] E[\underline{X3}] E [\underline{X4}']$$

$$= 1 - E[\underline{X1}] + E[\underline{X1}] E[\underline{X4}] + E[\underline{X1}] E[\underline{X2}] E[\underline{X3}]$$

$$- E[\underline{X1}] E[\underline{X2}] E[\underline{X3}] E[\underline{X4}]. \qquad \nabla \quad (2.41)$$

In summary, provided the inputs to a logic block are statistically independent and the signals experience a pure delay (i.e., $\underline{\tau}_r = \underline{\tau}_f = \underline{\tau}$), the expected value of the output from the logic block in Fig. 2.3, $E[\underline{Z}(t)]$, is readily obtained. First, $E[\underline{z}(t)]$, the output from the ideal logic circuit is evaluated by expressing $\underline{z}(t)$ as a sum of nonoverlapping implicants and then using the probability expression theorem. Finally, $E[\underline{Z}(t)]$ is obtained from $E[\underline{z}(t)]$ by convolving $E[\underline{z}(t)]$ with the p.d.f. of $\underline{\tau}$, $f_{\underline{\tau}}(t)$, as derived in Eq.(2.6).

2.3 Discriminating Delay ($\tau_r \neq \tau_f$)

As pointed out in Chapter 1, the delays τ_r experienced by the rise transitions typically do not equal the delays τ_f of the fall transitions. This situation can be analyzed by decomposing each 0,1 binary signal into two counting signals, as illustrated in Fig.1.6. $x^+(t)$ is referred to as the rise counting signal while $x^-(t)$ is called the fall counting signal.

A typical logic block with $\tau_r \neq \tau_f$ is illustrated in Fig. 2.5. Again, for ease of discussion, a single output $\underline{Z}(t)$ is considered. The input to the discriminating delay element is

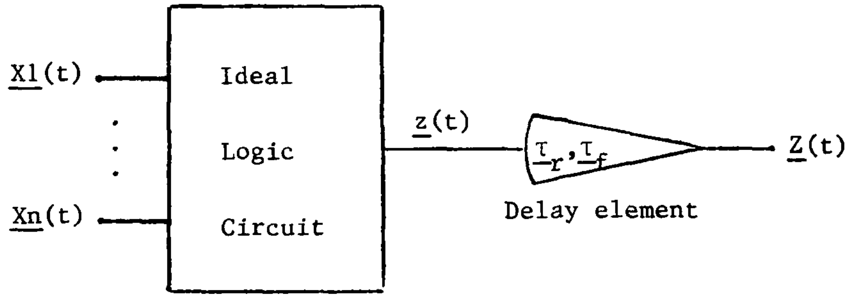


Fig. 2.5 A typical logic block with $\tau_r \neq \tau_f$.

$$\underline{z}(t) = \underline{z}^+(t) - \underline{z}^-(t) \ . \tag{2.42}$$

By definition, the output of the discriminating delay element is

$$\underline{Z}(t) = \underline{z}^+(t-\underline{\tau}_r) - \underline{z}^-(t-\underline{\tau}_f). \tag{2.43}$$

Because the delay element only delays the rising and falling transitions of $\underline{z}(t)$, it follows that

$$\underline{Z}^+(t) = \underline{z}^+(t-\underline{\tau}_r) \tag{2.44}$$

$$\underline{Z}^-(t) = \underline{z}^-(t-\underline{\tau}_f).$$

Therefore, Eq.(2.43) may also be expressed as

$$\underline{Z}(t) = \underline{Z}^+(t) - \underline{Z}^-(t). \tag{2.45}$$

Of interest is $E[\underline{Z}(t)]$. Unfortunately, a straightforward method for directly obtaining $\underline{z}^+(t)$ and $\underline{z}^-(t)$ given the counting signals for

$\underline{X1}(t),\ldots,\underline{Xn}(t)$ is not known. Difficulties arise because the counting signals are not binary. The problem is solved by considering time derivatives of the counting signal expectations. Later in this section a method is derived for obtaining $\dot{E}[\underline{z}^+(t)]$ and $\dot{E}[\underline{z}^-(t)]$ where the dot denotes differentiation with respect to time. For now, it is shown how to obtain $E[\underline{Z}(t)]$ from $\dot{E}[\underline{z}^+(t)]$ and $\dot{E}[\underline{z}^-(t)]$.

By examination of Eq.(2.43), observe that

$$E[\underline{Z}(t)] = E[\underline{z}^+(t-\underline{\tau}_r)] - E[\underline{z}^-(t-\underline{\tau}_f)] \tag{2.46}$$

where, by the conditional expectation theorem [18, p.208],

$$\begin{aligned} E[\underline{z}^+(t-\underline{\tau}_r)] &= E_{\underline{\tau}_r,\underline{z}^+(t)}[\underline{z}^+(t-\underline{\tau}_r)] \\ &= E_{\underline{\tau}_r}[E_{\underline{z}^+(t)}[\underline{z}^+(t-\underline{\tau}_r)|\underline{\tau}_r=\tau_r]] \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} E[\underline{z}^-(t-\underline{\tau}_f)] &= E_{\underline{\tau}_f,\underline{z}^-(t)}[\underline{z}^-(t-\underline{\tau}_f)] \\ &= E_{\underline{\tau}_f}[E_{\underline{z}^-(t)}[\underline{z}^-(t-\underline{\tau}_f)|\underline{\tau}_f=\tau_f]]. \end{aligned} \tag{2.48}$$

Taking the time derivative of Eq.(2.46) and making use of Eqs.(2.47) and (2.48), it follows that

$$\begin{aligned} \dot{E}[\underline{Z}(t)] = {} & \frac{d}{dt} E_{\underline{\tau}_r}[E_{\underline{z}^+(t)}[\underline{z}^+(t-\underline{\tau}_r)|\underline{\tau}_r=\tau_r]] \\ & - \frac{d}{dt} E_{\underline{\tau}_f}[E_{\underline{z}^-(t)}[\underline{z}^-(t-\underline{\tau}_f)|\underline{\tau}_f=\tau_f]]. \end{aligned} \tag{2.49}$$

Let $f_{\underline{\tau}_r}(\tau_r)$ and $f_{\underline{\tau}_f}(\tau_f)$ denote the p.d.f.'s of $\underline{\tau}_r$ and $\underline{\tau}_f$, respectively. Carrying out the expectations with respect to $\underline{\tau}_r$ and $\underline{\tau}_f$, Eq.(2.49) becomes

$$\dot{E}[\underline{Z}(t)] = \frac{d}{dt} \int_0^\infty E[\underline{z}^+(t-\tau_r)] f_{\underline{\tau}_r}(\tau_r) d\tau_r$$

$$- \frac{d}{dt} \int_0^\infty E[\underline{z}^-(t-\tau_f)] f_{\underline{\tau}_f}(\tau_f) d\tau_f$$

$$= \int_0^\infty \dot{E}[\underline{z}^+(t-\tau_r)] f_{\underline{\tau}_r}(\tau_r) d\tau_r$$

$$- \int_0^\infty \dot{E}[\underline{z}^-(t-\tau_f)] f_{\underline{\tau}_f}(\tau_f) d\tau_f$$

$$= \dot{E}[\underline{z}^+(t)] * f_{\underline{\tau}_r}(t) - \dot{E}[\underline{z}^-(t)] * f_{\underline{\tau}_f}(t). \tag{2.50}$$

For convenience, let t = 0 be the time reference. Integrating Eq.(2.50) yields

$$E[\underline{Z}(t)] = E[\underline{Z}(0)] + \int_0^t \{\dot{E}[\underline{z}^+(\theta)] * f_{\underline{\tau}_r}(\theta) - \dot{E}[\underline{z}^-(\theta)] * f_{\underline{\tau}_f}(\theta)\} d\theta. \tag{2.51}$$

Note that Eq.(2.51) expresses $E[\underline{Z}(t)]$ in terms of the time derivatives of the counting signal expectations at the delay element input.

Having derived Eq.(2.51), attention is now focused on obtaining

$\dot{E}[\underline{z}^+(t)]$ and $\dot{E}[\underline{z}^-(t)]$. In the derivation of the probability expression theorem, recall that Boolean orthogonalization was used to express the switching function for $\underline{z}(t)$ in the sum of nonoverlapping implicants form. This form leads to another useful result. After Boolean orthogonalization, the switching function is of the form

$$\underline{z} = g_o(\underline{X1},\ldots,\underline{Xn}) = \underline{z}_\ell = \underline{I}_{10} \vee \underline{I}_{11} \vee \ldots \vee \underline{I}_{1(\ell-1)} \tag{2.52}$$

Because the terms are nonoverlapping, only one implicant is nonzero at any instant. As a result, an equivalent expression for the switching function is obtained by replacing each OR operation in Eq.(2.52) by arithmetic addition as shown below:

$$\underline{z} = \underline{I}_{10} + \underline{I}_{11} + \ldots + \underline{I}_{1(\ell-1)}. \tag{2.53}$$

In addition, each implicant is of the form

$$\underline{I}_{1k} = \underline{A}_{1k} \cdot \underline{A}_{2k} \cdot \ldots \cdot \underline{A}_{nk} \ ; \ k = 0,\ 1,\ldots,(\ell-1) \tag{2.54}$$

where

$$\underline{A}_{jk} \ \varepsilon \ (\underline{Xj},\ \underline{Xj}',1);\ j=1,\ldots,n. \tag{2.55}$$

Consequently, an equivalent expression for $\underline{I}_{1k}$ results when each AND operation in Eq.(2.54) is replaced by arithmetic multiplication as follows:

$$\underline{I}_{1k} = (\underline{A}_{1k})(\underline{A}_{2k})\ldots(\underline{A}_{nk}) \ ; \ k = 0,1,\ldots,(\ell-1). \tag{2.56}$$

The previous observations are summarized in the following theorem:

The Arithmetic Expression Theorem

Let $\underline{z} = g_o(\underline{X1},\ldots,\underline{Xn})$ be a switching expression of the n switching variables $\underline{X1},\ldots,\underline{Xn}$. If $\underline{z}$ is expressed as a sum of nonoverlapping implicants, then an equivalent arithmetic expression for $\underline{z}$ is obtained by interchanging each

1) OR operation, $\vee$, by the arithmetic operation of addition,
2) AND operation, $\cdot$, by the arithmetic operation of multiplication.

It should be emphasized that performing the arithmetic operations in the equivalent arithmetic expression always yields the correct binary values (i.e. 0 or 1) for $\underline{z}$.

Example 2.5 Consider the switching expression $z = g_o(X1, X2, X3) = (X1 \vee X2')\cdot X3$ whose truth table is given below.

X1	X2	X3	z
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A sum of nonoverlapping implicants form for z is

$$z = X1 \cdot X3 \vee X1' \cdot X2' \cdot X3. \tag{2.57}$$

By the arithmetic expression theorem, z can be written as

$$z = (X1)(X3) + (X1')(X2')(X3). \tag{2.58}$$

This arithmetic expression can be shown to be valid by checking each row of the truth table. E.g., for the sixth row in which X1=1, X2=0, X3=1, application of Eq.(2.58) yields z=(1)(1)+(0)(1)(1) = 1 as predicted by the truth table. ∇

Application of the arithmetic expression theorem results in an arithmetic form for $\underline{z}(t)$. In particular,

$$\underline{z}(t) = \sum_{k=0}^{\ell-1} \underline{I}_{lk}(t)$$

$$= \sum_{k=0}^{\ell-1} \prod_{j=1}^{n} \underline{A}_{jk}(t) \tag{2.59}$$

The time derivative of $\underline{z}(t)$, assuming $\underline{z}(t)$ is mean-square differentiable, is given by

$$\dot{\underline{z}}(t) = \sum_{k=0}^{\ell-1} \sum_{i=1}^{n} \dot{\underline{A}}_{ik}(t) \prod_{\substack{j=1 \\ j\neq i}}^{n} \underline{A}_{jk}(t) . \tag{2.60}$$

Recall from Eq. (2.55) that

$$\underline{A}_{ik} \in (\underline{Xi}, \underline{Xi}', 1). \tag{2.61}$$

When $\underline{A}_{ik} = \underline{Xi}$, $\dot{\underline{A}}_{ik} = \dot{\underline{Xi}}$. When $\underline{A}_{ik} = \underline{Xi}'$,

$$\dot{\underline{A}}_{ik} = \frac{d}{dt}[\underline{Xi}'] = -\dot{\underline{Xi}} \tag{2.62}$$

The latter result follows because a rise transition in $\underline{Xi}'$ corresponds to a fall transition in $\underline{Xi}$ and vice versa. Finally, when $\underline{A}_{ik} = 1$, $\dot{\underline{A}}_{ik} = 0$. Utilizing the above and collecting terms in $\dot{\underline{Xk}}$, Eq.(2.60) can be rewritten as

$$\dot{\underline{z}}(t) = \sum_{k=1}^{n} \dot{\underline{Xk}}(t)\underline{Bk}(t) \tag{2.63}$$

<u>Example 2.6</u> Given the switching function

$$\underline{z}(t) = \underline{X1}(t) \cdot \underline{X2}(t) \cdot \underline{X3}(t) \vee (\underline{X1}(t) \cdot \underline{X2}(t))' \cdot \underline{X3}'(t), \tag{2.64}$$

express $\dot{\underline{z}}$ (t) in the form of Eq.(2.63).

For convenience the time symbol t and the underbar symbol denoting random process are suppressed in the ensuing discussion whenever the meaning is clear. The first step is to express z as a sum of nonoverlapping implicants. This results in

$$z = X1 \cdot X2 \cdot X3 \vee X1' \cdot X2 \cdot X3' \vee X2' \cdot X3'. \tag{2.65}$$

Next the arithmetic expression theorem is applied to yield

$$z = (X1)(X2)(X3) + (X1')(X2)(X3') + (X2')(X3'). \tag{2.66}$$

The time derivative of Eq. (2.63) is

$$\dot{z} = (\dot{X}1)(X2)(X3) + (X1)(\dot{X}2)(X3) + (X1)(X2)(\dot{X}3)$$

$$+ (\frac{d}{dt}[X1'])(X2)(X3') + (X1')(\dot{X}2)(X3')$$

$$+ (X1')(X2)(\frac{d}{dt}[X3']) + (\frac{d}{dt}[X2'])(X3')$$

$$+ (X2')(\frac{d}{dt}[X3']). \qquad (2.67)$$

Substitution of Eq.(2.62) into Eq. (2.67) yields

$$\dot{z} = (\dot{X}1)(X2)(X3) + (X1)(\dot{X}2)(X3) + (X1)(X2)(\dot{X}3)$$

$$- (\dot{X}1)(X2)(X3') + (X1')(\dot{X}2)(X3') - (X1')(X2)(\dot{X}3)$$

$$- (\dot{X}2)(X3') - (X2')(\dot{X}3). \qquad (2.68)$$

Collecting terms result in

$$\dot{z} = (\dot{X}1)(X2)(X3 - X3') + (\dot{X}2)[(X1)(X3) + (X3')(X1'-1)]$$

$$+ (\dot{X}3)\ [(X2)(X1-X1') - (X2')] \qquad (2.69)$$

$$= (\dot{X}1)(B1) + (\dot{X}2)(B2) + (\dot{X}3)(B3)$$

where

$$B1 = (X2)(X3-X3')$$

$$B2 = (X1)(X3) + (X3')(X1'-1)$$

$$B3 = (X2)(X1-X1')-(X2') \qquad (2.70)$$

Since sample functions of $\underline{z}(t)$ are ideal 0,1 binary signals, sample functions of $\dot{\underline{z}}(t)$ consist entirely of positive and negative unit area impulses which occur at the rise and fall transition times of $\underline{z}(t)$, respectively. At each instant of time $\underline{Bk}(t)$ is either 0, 1, or -1. Typical sketches of sample functions from $\underline{z}(t)$ and $\dot{\underline{z}}(t)$ are presented in Fig. 2.6(a) and (b), respectively. Note that $\dot{\underline{z}}(t)$ consists entirely of unit area impulses with alternating polarity.

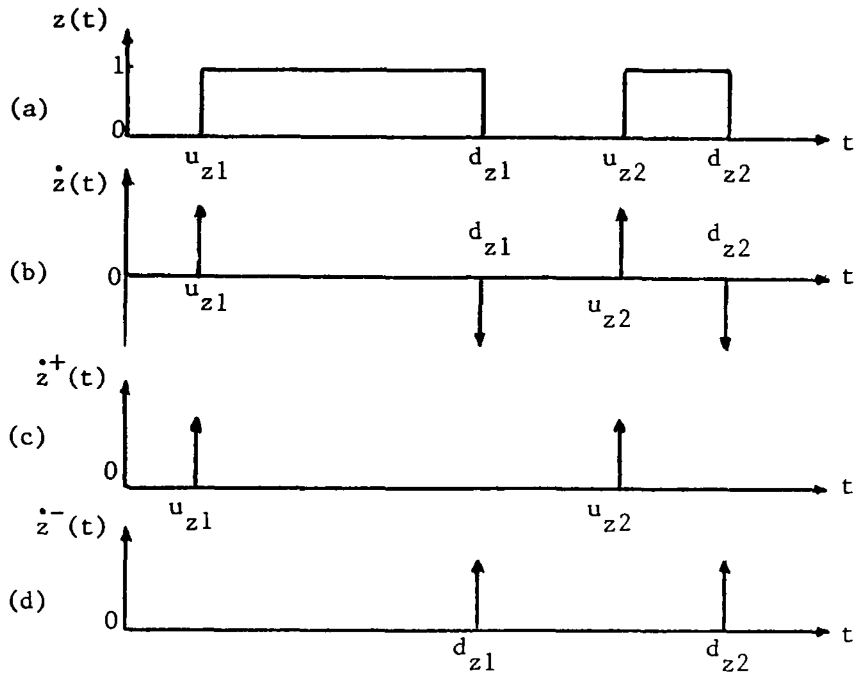


Fig. 2.6 Sample functions of a) $\underline{z}(t)$, b) $\dot{\underline{z}}(t)$, c) $\dot{\underline{z}}^+(t)$, d) $\dot{\underline{z}}^-(t)$

By definition, $\dot{z}^+(t)$ is the time derivative of $\underline{z}^+(t)$ and consists of all the positive area impulses in $\dot{\underline{z}}(t)$. Similarly, $\dot{\underline{z}}^-(t)$ is the time derivative of $\underline{z}^-(t)$. Recalling that $\underline{z}(t) = \underline{z}^+(t) - \underline{z}^-(t)$, $-\dot{\underline{z}}^-(t)$ consists of all the negative area impulses in $\dot{\underline{z}}(t)$. Typical sketches of sample functions from $\dot{\underline{z}}^+(t)$ and $\dot{\underline{z}}^-(t)$ are presented in Fig. 2.6 (c) and (d), respectively. Observe that

$$\dot{\underline{z}}(t) = \dot{\underline{z}}^+(t) - \dot{\underline{z}}^-(t). \tag{2.71}$$

In the analysis to follow it is necessary to have analytical expressions for $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$ in terms of the switching variables $\underline{Xk}$, k = 1,...,n, and their time derivatives. For this purpose, an equivalent arithmetic expression for $\underline{Bk}(t)$ must be obtained such that only one term in its equivalent expression is nonzero at each instant and the value of that nonzero term equals the value of $\underline{Bk}(t)$ whenever it is nonzero. This form is referred to as the arithmetic canonical form.

The arithmetic canonical form is obtained by tabulating as in a truth table, all possible combinations of the values for the variables in $\underline{Bk}$ (i.e.; 0,1), along with the corresponding value of $\underline{Bk}$ (i.e.; -1,0,1). Having this table, a product term is formed for each row by writing $\underline{Xi}$ as a factor if $\underline{Xi}$ = 1 and $\underline{Xi}'$ if $\underline{Xi}$ = 0. Denote the product term for the m^{th} row in the table by $\underline{Pk}(m)$. In forming the product terms in this manner, one and only one can take on the value 1 for a specified set of values for $\underline{Xk}$; k = 1,...,n. Specifically, $\underline{Pk}(m)$ = 1 only for the values of $\underline{Xk}$ specified in the m^{th} row of the table. It follows that the expression obtained by multiplying $\underline{Pk}(m)$ by the value of $\underline{Bk}$ corresponding to the m^{th} row and summing all such products results in the desired equivalent expression for $\underline{Bk}$. Denote the value of $\underline{Bk}$ corresponding to the m^{th} row of the table by $\underline{Bk}(m)$. The arithmetic canonical form for $\underline{Bk}$ is then given by

$$\underline{Bk} = \sum_{m=1}^{2^{\gamma}} \underline{Bk}(m)\ \underline{Pk}(m) \qquad (2.72)$$

where $\gamma \leq n-1$ is the number of input variables $\underline{Xk}$ upon which $\underline{Bk}$ depends.

Example 2.6 (cont.) Determine the arithmetic canonical forms for B1, B2, B3 in Eq. (2.70).

Following the procedure outlined above, the following tables result where an additional column has been included for tabulating Pk(m).

B1 = (X2)(X3-X3')				
m	X2	X3	B1(m)	P1(m)
1	0	0	0	(X2')(X3')
2	0	1	0	(X2')(X3)
3	1	0	-1	(X2)(X3')
4	1	1	1	(X2)(X3)

B2 = (X1)(X3)+(X3')(X1'-1)				
m	X1	X3	B2(m)	P2(m)
1	0	0	0	(X1')(X3')
2	0	1	0	(X2')(X3)
3	1	0	-1	(X1)(X3')
4	1	1	1	(X1)(X3)

B3 = (X2)(X1-X1')-(X2')				
m	X1	X3	B3(m)	P3(m)
1	0	0	-1	(X1')(X2')
2	0	1	-1	(X1')(X2)
3	1	0	-1	(X1)(X2')
4	1	1	1	(X1)(X2)

Using Eq. (2.72), the arithmetic canonical forms for the coefficients Bk become

B1 = (0)(X2')(X3')+(0)(X2')(X3)+(-1)(X2)(X3')+(1)(X2)(X3)=(X2)(X3)-(X2)(X3')

B2 = (0)(X1')(X3')+(0)(X1')(X3)+(-1)(X1)(X3')+(1)(X1)(X3)=(X1)(X3)-(X1)(X3')

B3 = (-1)(X1')(X2')+(-1)(X1')(X2)+(-1)(X1)(X2')+(1)(X1(X2)

= (X1)(X2)-(X1)(X2')-(X1')(X2)-(X1')(X2'). ∇ (2.73)

In the arithmetic canonical form for $\underline{Bk}(t)$ given by Eq.(2.72), the terms involved in the summation are either nonnegative or nonpositive. Denote the sum of all nonnegative terms by $\underline{Bk}^{+}(t)$ and the sum of all nonpositive terms by $-\underline{Bk}^{-}(t)$. Then

$$\underline{Bk}(t) = \underline{Bk}^{+}(t)-\underline{Bk}^{-}(t). \tag{2.74}$$

Note that

$$\underline{Bk}^{+}(t) \geq 0 \quad \text{and} \quad \underline{Bk}^{-}(t) \geq 0. \tag{2.75}$$

Example 2.6 (cont.). Separate Bk in Eq.(2.73) into Bk^+ and Bk^-; k = 1,2,3. By inspection of Eq.(2.73),

$$B1^+ = (X2)(X3)\ ,\ B1^- = (X2)(X3')$$

$$B2^+ = (X1)(X3)\ ,\ B2^- = (X1)(X3')$$

$$B3^+ = (X1)(X2)\ ,\ B3^- = (X1')(X2)+(X1)(X2')+(X1')(X2'). \quad \nabla \quad (2.76)$$

Because the 0,1 binary random process $\underline{Xk}(t)$ equals $\underline{Xk}^+(t)-\underline{Xk}^-(t)$, recall that its time derivative is given by

$$\dot{\underline{Xk}}\ (t) = \dot{\underline{Xk}}^+(t) - \dot{\underline{Xk}}^-(t) \quad (2.77)$$

where both $\dot{\underline{Xk}}(t)$ and $\dot{\underline{Xk}}^-(t)$ consist entirely of postive unit area impulses. Substitution of Eqs.(2.77) and (2.74) into Eq.(2.63) yields

$$\dot{\underline{z}}(t) = \sum_{k=1}^{n} [\dot{\underline{Xk}}^+(t) - \dot{\underline{Xk}}^-(t)][\underline{Bk}^+(t) - \underline{Bk}^-(t)]$$

$$= \sum_{k=1}^{n} [\dot{\underline{Xk}}^+(t)\ \underline{Bk}^+(t) + \dot{\underline{Xk}}^-(t)\underline{Bk}^-(t)]$$

$$- \sum_{k=1}^{n} [\dot{\underline{Xk}}^+(t)\underline{Bk}^-(t) + \dot{\underline{Xk}}^-(t)\underline{Bk}^+(t)]. \quad (2.78)$$

Upon integration of Eq.(2.78) the first summation contributes to the rise transitions in $\underline{z}(t)$ while the second summation contributes to the fall transitions in $\underline{z}(t)$. By comparison of Eq.(2.78) with Eq.(2.71), it follows

that

$$\dot{\underline{z}}^{+}(t) = \sum_{k=1}^{n} [\dot{\underline{Xk}}^{+}(t)\underline{Bk}^{+}(t) + \dot{\underline{Xk}}^{-}(t)\ \underline{Bk}^{-}(t)]$$

$$\dot{z}^{-}(t) = \sum_{k=1}^{n} [\dot{\underline{Xk}}^{+}(t)\underline{Bk}^{-}(t) + \dot{\underline{Xk}}^{-}(t)\underline{Bk}^{+}(t)] \tag{2.79}$$

Example 2.6 (cont.) Determine analytical expressions for $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$ in terms of the switching variable for the switching function given by Eq.(2.64).

By direct substitution of Eqs.(2.76) into Eqs.(2.79), one obtains

$$\dot{z}^{+} = (\dot{X1}^{+})(X2)(X3) + (\dot{X1}^{-})(X2)(X3') + (\dot{X2}^{+})(X1)(X3)+(\dot{X2}^{-})(X1)(X3')$$

$$+ (\dot{X3}^{+})(X1)(X2) + (\dot{X3}^{-})[(X1')(X2) + (X1)(X2') + (X1')(X2')].$$

$$\dot{z}^{-} = (\dot{X1}^{+})(X2)(X3') +(\dot{X1}^{-})(X2)(X3) + (\dot{X2}^{+})(X1)(X3') + (\dot{X2}^{-})(X1)(X3)$$

$$+ (\dot{X3}^{+})[(X1')(X2) + (X1)(X2') + (X1')(X2')]+(\dot{X3}^{-})(X1)(X2)$$

$$\nabla \quad (2.80)$$

In summary, it is necessary to perform the following steps in order to obtain analytical expressions for $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$ in terms of the switching variables and their time derivatives:

1) Express $\underline{z}(t)$ as a sum of nonoverlapping implicants by means of the Boolean orthogonalization procedure.

2) Obtain an arithmetic expression for $\underline{z}(t)$ by means of the arithmetic expression theorem.

3) Differentiate with respect to time the arithmetic expression obtained in step (2).

4) Use Eq.(2.62) to eliminate the time derivatives of all complemented variables and collect terms in $\underline{\dot{X}k}$ to yield $\underline{\dot{z}}(t)$ in the form of Eq.(2.63).

5) Express each coefficient $\underline{Bk}$ by its arithmetic canonical form as given in Eq.(2.72).

6) Separate $\underline{Bk}$ into $\underline{Bk}^+$ and $\underline{Bk}^-$ and $\underline{\dot{X}k}$ into $\underline{\dot{X}k}^+$ and $\underline{\dot{X}k}^-$.

7) Use Eqs.(2.79) to obtain $\underline{\dot{z}}^+(t)$ and $\underline{\dot{z}}^-(t)$.

Having determined analytical expressions for $\underline{\dot{z}}^+(t)$ and $\underline{\dot{z}}^-(t)$, it is now possible to obtain expressions for $\dot{E}[\underline{z}^+(t)]$ and $\dot{E}[\underline{z}^-(t)]$ which are needed in the evaluation of Eq.(2.51). For any process $\underline{Y}(t)$ which is mean-square differentiable, it follows that [18, Sec. 9.6]

$$E[\underline{\dot{Y}}(t)] = \frac{d}{dt} E[\underline{Y}(t)] = \dot{E}[\underline{Y}(t)]. \tag{2.81}$$

Recall that $\underline{Bk}$ does not contain $\underline{Xk}$ as a factor. Assuming all of the input switching varibles to be statistically independent, $\underline{Bk}^+(t)$ is statistically independent of $\underline{\dot{X}k}^+(t)$ and $\underline{\dot{X}k}^-(t)$. Similarly, $\underline{Bk}^-(t)$ is also statistically independent of $\underline{\dot{X}k}^+(t)$ and $\underline{\dot{X}k}^-(t)$. Using this independence and the property stated in Eq.(2.81), the expected values of Eqs.(2.79) become

$$\dot{E}[\underline{z}^+(t)] = \sum_{k=1}^{n} \{\dot{E}[\underline{Xk}^+(t)]E[\underline{Bk}^+(t)] + \dot{E}[\underline{Xk}^-(t)]\ E\ [\underline{Bk}^-(t)]\}$$

$$\dot{E}[\underline{z}^-(t)] = \sum_{k=1}^{n} \{\dot{E}[\underline{Xk}^+(t)]E[\underline{Bk}^-(t)] + \dot{E}[\underline{Xk}^-(t)]E[\underline{Bk}^+(t)]\} \tag{2.82}$$

Removal of expected values of complemented variables in Eq.(2.82) can be accomplished by using Eq.(2.38) which states that

$$E[\underline{Xk}'(t)] = 1 - E[\underline{Xk}(t)]. \qquad (2.83)$$

Example 2.6 (cont.) Utilizing Eqs.(2.82) and (2.83), obtain the expected values of $\dot{\underline{z}}^+$ and $\dot{\underline{z}}^-$ specified by Eqs.(2.80). Assume all switching variables to be statistically independent.

By inspection, it follows that

$$\begin{aligned}\dot{E}[\underline{z}^+(t)] = {} & \dot{E}[\underline{X1}^+(t)]E[\underline{X2}(t)]E[\underline{X3}(t)] + \dot{E}[\underline{X1}^-(t)]E[\underline{X2}(t)]\{1-E[\underline{X3}(t)]\} \\ & + \dot{E}[\underline{X2}^+(t)]E[\underline{X1}(t)]E[\underline{X3}(t)] + \dot{E}[\underline{X2}^-(t)]E[\underline{X1}(t)]\{1-E[\underline{X3}(t)]\} \\ & + \dot{E}[\underline{X3}^+(t)]E[\underline{X1}(t)]E[\underline{X2}(t)] + \dot{E}[\underline{X3}^-(t)]\ \{E[\underline{X2}(t)]\ \{1 - E[\underline{X1}(t)]\} \\ & + E[\underline{X1}(t)]\{1 - E[\underline{X2}(t)]\} + \{1 - E[\underline{X1}(t)]\}\ \{1 - E[\underline{X2}(t)]\}\}\end{aligned}$$

$$\begin{aligned}\dot{E}[\underline{z}^-(t)] = {} & \dot{E}[\underline{X1}^+(t)]E[\underline{X2}(t)]\{1 - E[\underline{X3}(t)]\} + \dot{E}[\underline{X1}^-(t)]E[\underline{X2}(t)]E[\underline{X3}(t)] \\ & + \dot{E}[\underline{X2}^+(t)]E[\underline{X1}(t)]\{1 - E[\underline{X3}(t)\} + \dot{E}[\underline{X2}^-(t)]E[\underline{X1}(t)]E[\underline{X3}(t)] \\ & + \dot{E}[\underline{X3}^+(t)]\{E[\underline{X2}(t)]\{1 - E[\underline{X1}(t)]\} + E[\underline{X1}(t)]\{1 - E[\underline{X2}(t)]\} \\ & + \{1 - E[\underline{X1}(t)]\}\{1 - E[\underline{X2}(t)]\}\} + \dot{E}[\underline{X3}^-(t)]E[\underline{X1}(t)]E[\underline{X2}(t)]\end{aligned}$$

$$\nabla \qquad (2.84)$$

The previous discussion has supplied all of the ingredients necessary for the evaluation of Eq.(2.51) which is repeated below.

$$E[\underline{Z}(t)] = E[\underline{Z}(0)] + \int_0^t \{\dot{E}[\underline{z}^+(\theta)]*f_{\underline{\tau}_r}(\theta) - \dot{E}[\underline{z}^-(\theta)]*f_{\underline{\tau}_f}(\theta)\}\,d\theta \qquad (2.85)$$

As a final remark, recall that $\underline{Z}(t)$ is a 0,1 binary random process and that it can be decomposed into counting processes as in Eq.(2.45). It follows that

$$\dot{\underline{Z}}(t) = \dot{\underline{Z}}^{+}(t) - \dot{\underline{Z}}^{-}(t) \,. \tag{2.86}$$

The expectation of Eq.(2.86) is given by

$$E[\dot{\underline{Z}}(t)] = E[\dot{\underline{Z}}^{+}(t)] - E[\dot{\underline{Z}}^{-}(t)]. \tag{2.87}$$

Utilizing the interchangeability of differentiation and expectation, as stated in Eq.(2.81),

$$\dot{E}[\underline{Z}(t)] = \dot{E}[\underline{Z}^{+}(t)] - \dot{E}[\underline{Z}(t)]. \tag{2.88}$$

Taking the time derivative of Eq.(2.85), there results

$$\dot{E}[\underline{Z}(t)] = \dot{E}[\underline{z}^{+}(t)]*f_{\underline{\tau}_r}(t) - \dot{E}[\underline{z}^{-}(t)]*f_{\underline{\tau}_f}(t) \,. \tag{2.89}$$

Because $\dot{E}[\underline{Z}^{+}(t)]$, $\dot{E}[\underline{Z}^{-}(t)]$, $\dot{E}[\underline{z}^{+}(t)]$, $\dot{E}[\underline{z}^{-}(t)]$, $f_{\underline{\tau}_r}(t)$, $f_{\underline{\tau}_f}(t)$ are each nonnegative, it follows that

$$\begin{aligned} \dot{E}[\underline{Z}^{+}(t)] &= \dot{E}[\underline{z}^{+}(t)] * f_{\underline{\tau}_r}(t) \\ \dot{E}[\underline{Z}^{-}(t)] &= \dot{E}[\underline{z}^{-}(t)]*f_{\underline{\tau}_f}(t) \,. \end{aligned} \tag{2.90}$$

Example 2.7 Consider the physical NAND gate whose model and inputs are shown in Fig. 2.7. The inputs are deterministic and are assumed

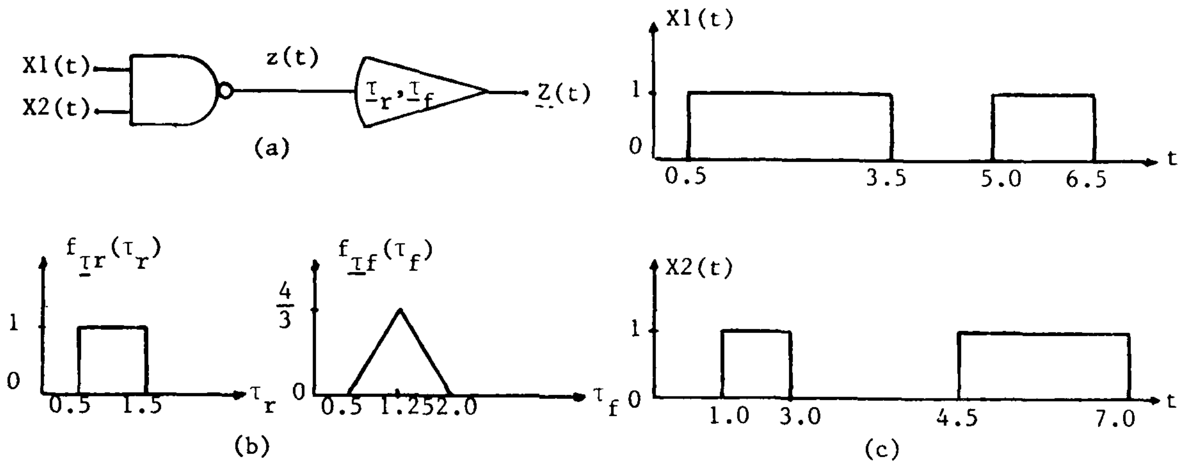


Fig. 2.7(a) Circuit model (b) p.d.f.'s of $\underline{\tau}_r$ and $\underline{\tau}_f$ (c) input signals

to be zero everywhere except for the pulses shown. In particular $X1(t) = X2(t) = 0$ for $t \leq 0$. As a result, $z(t) = 1$ for $t \leq 0$ and

$$E[\underline{Z}(0)] = \underline{Z}(0) = 1 \ . \tag{2.91}$$

It is desired to compute the output expected value $E[\underline{Z}(t)]$, using Eq.(2.85). Following the methods of this chapter, the output expected value is computed in two steps. First, $\dot{E}[z^+(t)]$ and $\dot{E}[z^-(t)]$ are computed. The convolutions are then evaluated. Note that

$$z(t) = (X1(t) \cdot X2(t))' = X1'(t) \vee X2'(t). \tag{2.92}$$

In terms of nonoverlapping implicants, $z(t)$ becomes

$$z(t) = X1'(t) \vee X1(t) \cdot X2'(t). \tag{2.93}$$

The arithmetic expression for z(t) is given by

$$z(t) = X1'(t) + X1(t) \quad X2'(t). \tag{2.94}$$

Differentiation with respect to time yields

$$\begin{aligned} \dot{z}(t) &= \dot{X}1'(t) + \dot{X}1(t)X2'(t) + X1(t)\dot{X}2'(t) \\ &= -\dot{X}1(t)+\dot{X}1(t)X2'(t) - X1(t)\dot{X}2(t) \\ &= -\dot{X}1(t)(1 - X2'(t)) - X1(t)\dot{X}2(t) \\ &= -\dot{X}1(t)X2(t) - X1(t)X2(t). \end{aligned} \tag{2.95}$$

Since the coefficients in Eq. (2.95) are in canonical form,

$$B1^{+} = 0,\ B1^{-} = X2,\ B2^{+} = 0,\ B2^{-} = X1. \tag{2.96}$$

Using eqs. (2.79),

$$\dot{z}^{+}(t) = \dot{X}1^{-}(t)\ X2(t) + \dot{X}2^{-}(t)X1(t) \tag{2.97}$$

$$\dot{z}^{-}(t) = \dot{X}1^{+}(t)\ X2(t)+\dot{X}2^{+}(t)\ X1(t).$$

Because the input signals are deterministic, $E[Xk(t)] = Xk(t)$.

Equations (2.97) are shown graphically in Fig. 2.8. In order to obtain, $\dot{E}[\underline{Z}(t)]$, the derivative of the expectation of the delayed output, convolutions are performed as in Eq.(2.89).

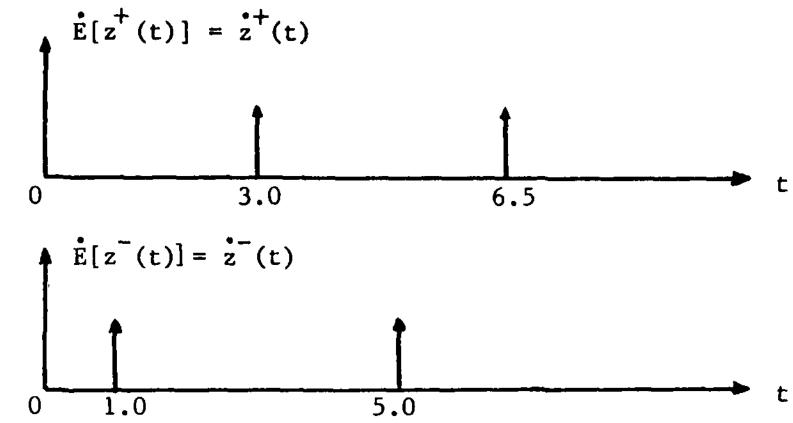


Fig. 2.8 Derivatives of the counting processes of z(t).

The two terms of Eq. (2.89) are shown in Fig. 2.9.

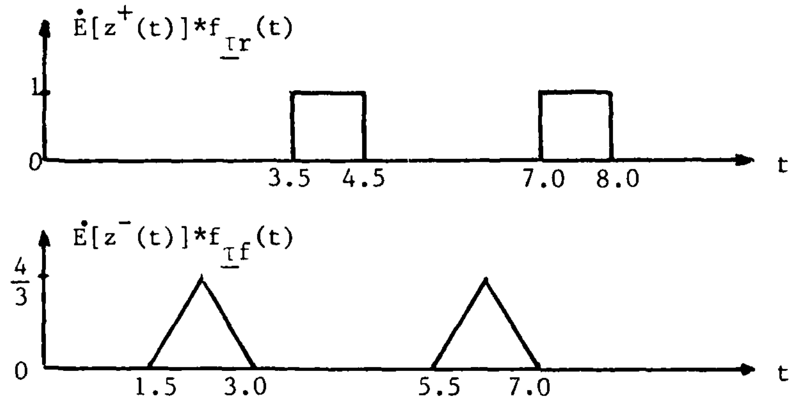


Fig. 2.9 Sketches of the two terms in Eq.(2.99).

Using the results of Fig. 2.9 in Eq.(2.85), the integration yields the two terms sketched in Fig. 2.10. The inital condition is specified by Eq.(2.91). Performing the summations in Eq.(2.85), the final result for $E[\underline{Z}(t)]$ is shown in Fig. 2.11. As expected, $E[\underline{Z}(t)]$ is nonnegative and varies between 0 and 1. To evaluate system performance, it is

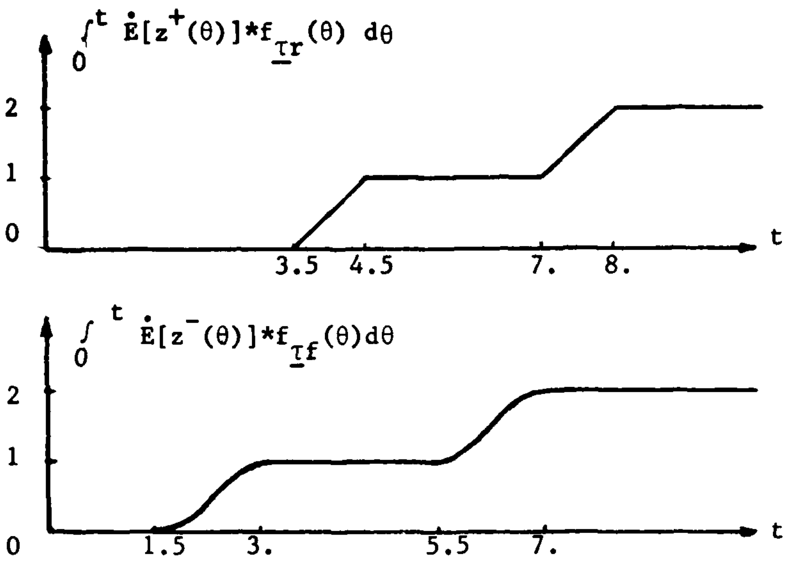


Fig. 2.10 Terms emerging from the integration in Eq.(2.85)

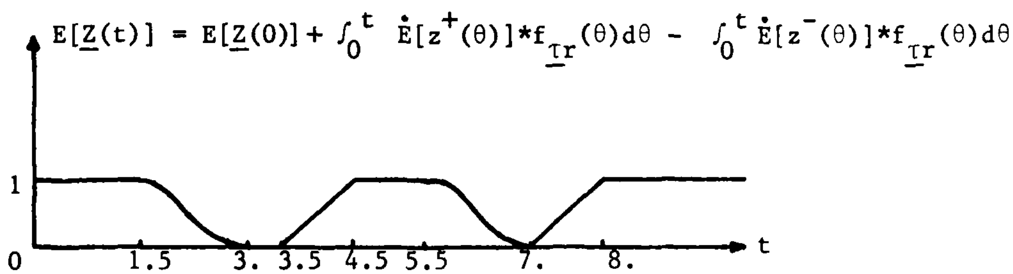


Fig. 2.11 Output expected value for example 2.7

interesting to compare $E[\underline{Z}(t)]$ with the predicted output $\hat{Z}(t)$ for the deterministic case in which τ_r and τ_f are specified to be $\tau_r = 1$ and $\tau_f = 1.25$. Since the average values of $\underline{\tau}_r$ and $\underline{\tau}_f$ have been selected, this is a best case situation which results in a minimum peak error of 0.5. The waveform for $\hat{Z}(t)$ is shown in Fig. 2.12. Recall from Eqs.(1.4) that

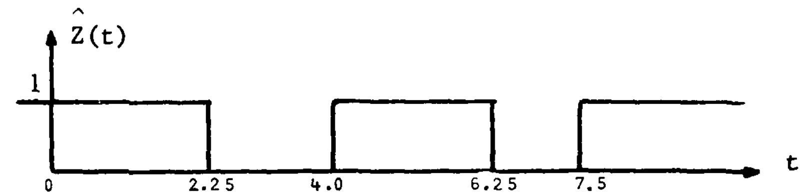


Fig. 2.12 $\hat{Z}(t)$ when $\tau_r = 1$ and $\tau_f = 1.25$

$$\Pr\{\text{error} \mid \hat{Z}(t_1) = 0\} = E[\underline{Z}(t_1)]$$

$$\Pr\{\text{error} \mid \hat{Z}(t_2) = 1\} = 1 - E[\underline{Z}(t_2)]. \quad (2.98)$$

It follows that Pr $\{\text{error} \mid \hat{Z}(t)\}$ for the results shown in Figs.2.11 and 2.12 is the waveform given in Fig. 2.13. Note that the error does not exceed 0.5. Errors larger than 0.5 will be encountered when values differing from $\tau_r = 1$, $\tau_f = 1.25$ are specified.

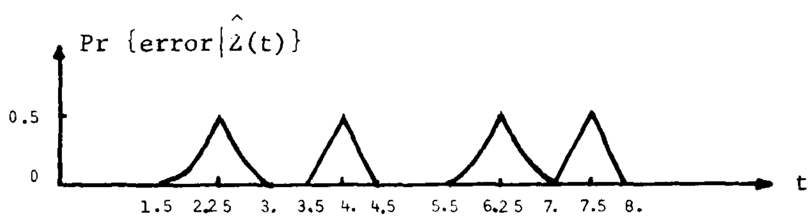


Fig. 2.13 Conditional error when $\tau_r = 1$, $\tau_f = 1.25$.

2.4 Tree-like networks

Thus far, the discussion has dealt with analysis of a single logic block. Given the expected value of each input signal and assuming the inputs to be statistically independent it was shown how to determine the

expected value of the output. Now consider a tree-like combinational network of logic blocks. A tree-like network was defined to be a network in which no more than one path exists from each input to every output. As a consequence, there is no more than one path from each input to any point of the network. Given that the inputs to the network are statistically independent, it follows that the inputs to each logic block within the network will also be statistically independent. Also, the expected values of the outputs of preceding logic blocks provide the expected values of the inputs to the following logic blocks. Therefore, by repeated applications of the methods developed in this chapter, it is possible to extend the analysis of a single logic block to the analysis of an entire tree-like combinational network. The extension is demonstrated in the following examples.

Example 2.8 A tree-like combinational network, whose logic blocks are assumed to include pure delay elements, is shown in Fig. 2.14 along with p.d.f.'s of the delays. For clarity, an upper case letter is used as a subscript to the symbol $\underline{\tau}$ of the delay to denote the gate type. In particular, $\underline{\tau}_A$ is the pure delay associated with AND gates, $\underline{\tau}_I$ is the pure delay associated with INVERTERS, and $\underline{\tau}_O$ is the pure delay associated with OR gates. All inputs are deterministic and depicted in Fig. 2.15. It is desired to evaluate the output expected value, $E[\underline{Z}(t)]$.

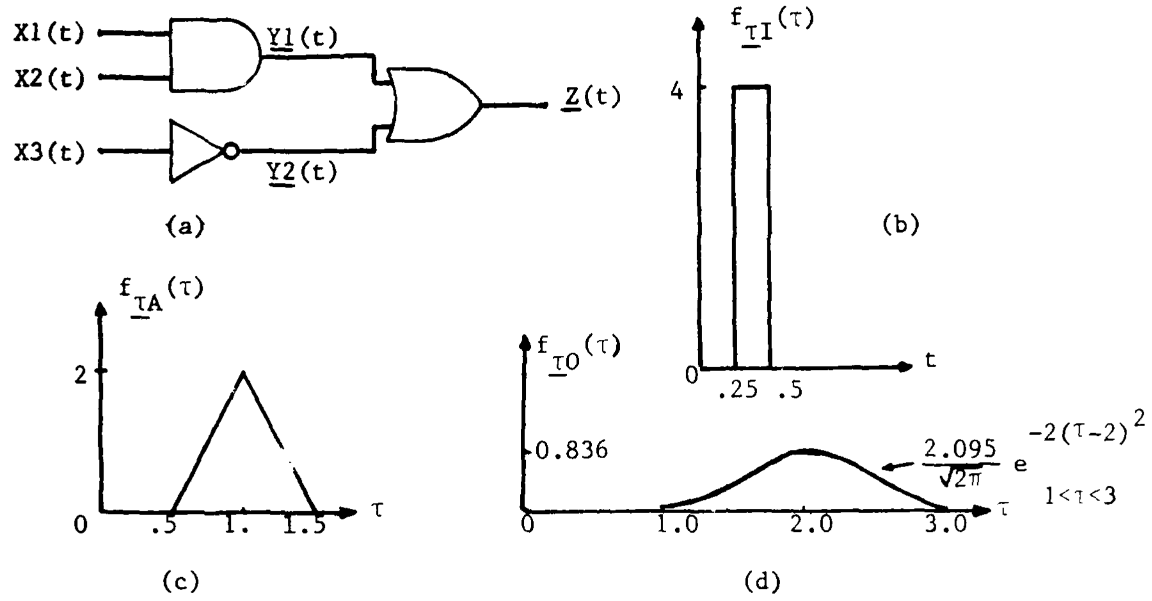


Fig. 2.14 (a) network for Ex. 2.8, (b) p.d.f of $\underline{\tau}_I$, (c) p.d.f. of $\underline{\tau}_A$,

(d) p.d.f of $\underline{\tau}_0$

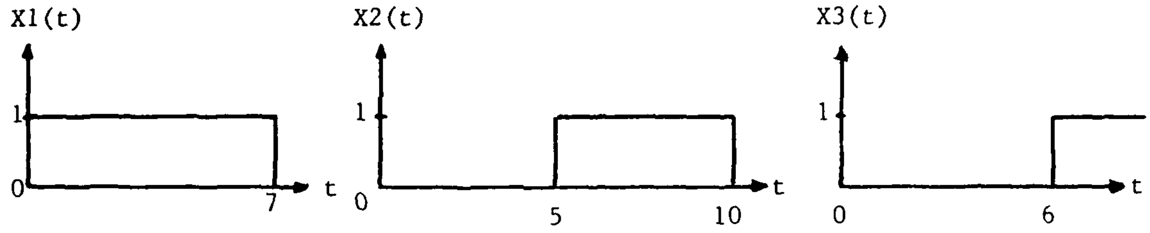


Fig. 2.15 Deterministic input signals to the network in Fig. 2.14(a)

The notation used for the signals in this example is identical to that illustrated in Fig. 2.1. Beginning at the network input and successively applying the methods derived in Sec. 2.2 results in the expected value of the network output. The ideal logic outputs for the AND and INVERTER gates are

$$y1(t) = X1(t)\cdot X2(t)$$
$$y2(t) = X3'(t) \; . \tag{2.99}$$

These are sketched in Fig. 2.16.

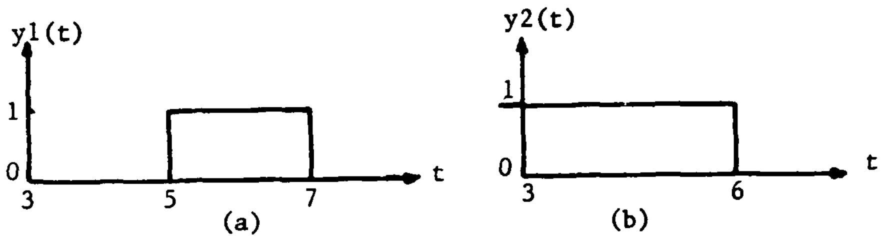


Fig. 2.16 Sketches of (a) y1(t) and (b) y2(t).

Using Eq.(2.6), the output expected values of the AND and INVERTER gates are

$$E[\underline{Y1}(t)] = y1(t)*f_{\underline{\tau}A}(t)$$

$$E[\underline{Y2}(t)] = y2(t)*f_{\underline{\tau}I}(t). \tag{2.100}$$

These are sketched in Fig. 2.17.

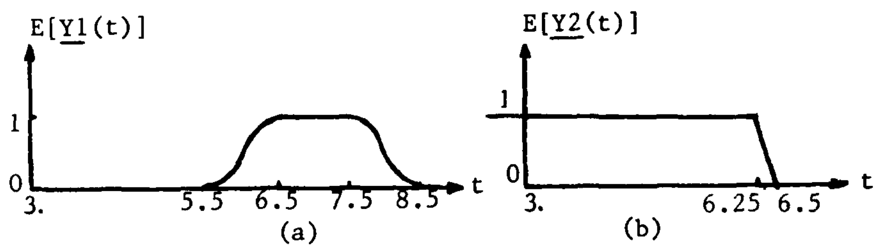


Fig. 2.17 Sketches of (a) $E[\underline{Y1}(t)]$ and (b) $E[\underline{Y2}(t)]$.

The OR gate is considered next. Since $\underline{Y1}(t)$ and $\underline{Y2}(t)$ are independent, it follows that the OR gate ideal logic output expected value is given by

$$E[\underline{z}(t)] = E[\underline{Y1}(t) \vee \underline{Y2}(t)] = E[\underline{Y1}(t)] + E[\underline{Y2}(t)]-E[\underline{Y1}(t)]E[\underline{Y2}(t)]. \tag{2.101}$$

This is illustrated in Fig. 2.18.

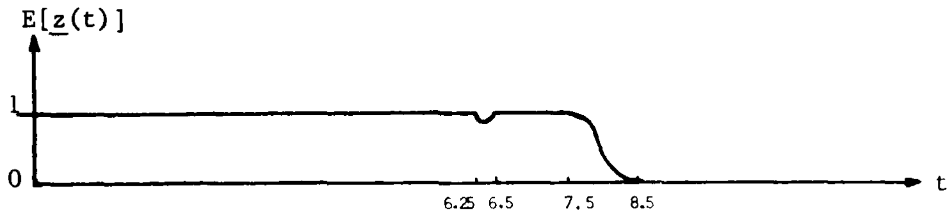


Fig. 2.18 Sketch of $E[\underline{z}(t)]$.

Finally, the expected value of the network output is

$$E[\underline{Z}(t)] = E[\underline{z}(t)]*f_{\underline{\tau}\,0}(t). \tag{2.102}$$

This is sketched in Fig. 2.19.

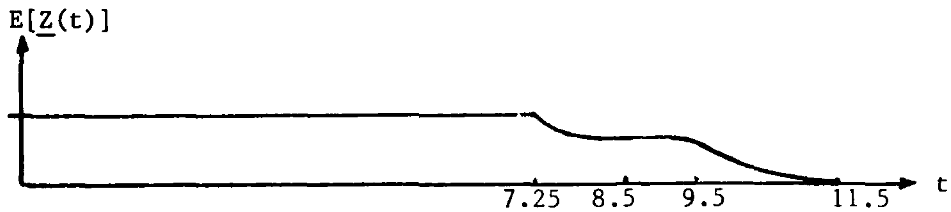


Fig. 2.19 Sketch of $E[\underline{Z}(t)]$

For performance evaluation, the expectation $E[\underline{Z}(t)]$ is compared with the predicted output $\hat{Z}(t)$, corresponding to the delays $\tau_I = 0.375$, $\tau_A = 1.0$, and $\tau_O = 2.0$, which were selected to yield minimal peak error. $\hat{Z}(t)$ is shown in Fig. 2.20.

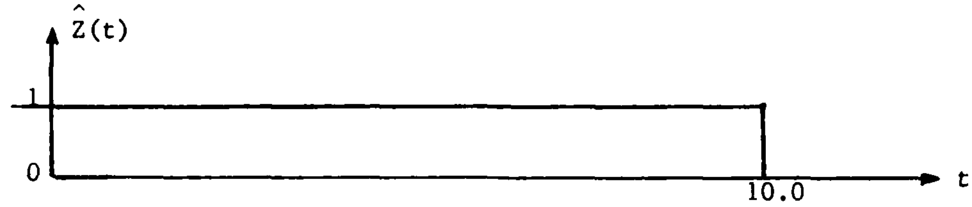


Fig. 2.20 The predicted signal $\hat{Z}(t)$, when $\tau_I = 0.375$, $\tau_A = 1.0$, and $\tau_O = 2.0$

The conditional probability of error, for this case $\Pr\{\text{error} \mid \hat{Z}(t)\}$, is sketched in Fig. 2.21.

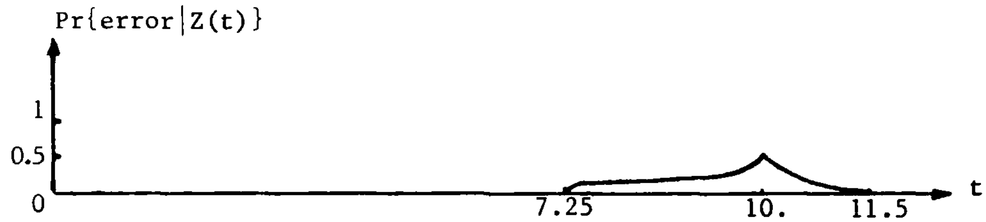


Fig. 2.21 Sketch of the conditional error when $\tau_I = 0.375$, $\tau_A = 1.0$, and $\tau_O = 2.0$

∇

Example 2.9. A tree-like combinational network is shown in Fig. 2.22. All logic gates are assumed to include discriminating delay elements and the network inputs are assumed to be statistically independent. Also assume the initial conditions $E[\underline{Y1}(0)]$, $E[\underline{Y2}(0)]$, and $E[\underline{Z}(0)]$ are known. It is desired to evaluate the output expected value, $E[\underline{Z}(t)]$, in terms of the input expectations and the p.d.f.'s of the gate propagation delays.

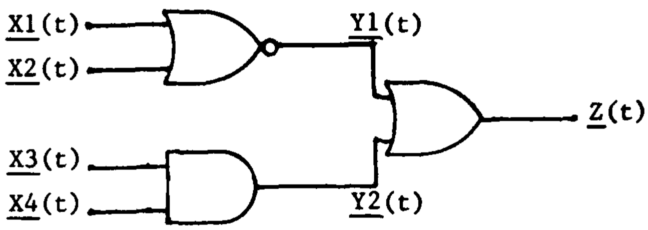


Fig. 2.22. Tree-like combinational circuit for Ex. 2.9

Once again, the notation of Fig. 2.1 is employed. Using the procedure of Sec. 2.3, the derivatives $\dot{E}[\underline{y1}^+(t)]$, $\dot{E}[\underline{y1}^-(t)]$, $\dot{E}[\underline{y2}^+(t)]$, and $\dot{E}[\underline{y2}^-(t)]$ are obtained as follows:

$$\dot{E}[\underline{y1}^+(t)] = \dot{E}[\underline{X1}^-(t)]\{1-E[\underline{X2}(t)]\} + \dot{E}[\underline{X2}^-(t)]\{1 - E[\underline{X1}(t)]\} \tag{2.103}$$

$$\dot{E}[\underline{y1}^-(t)] = \dot{E}[\underline{X1}^+(t)]\{1 - E[\underline{X2}(t)]\} + \dot{E}[\underline{X2}^+(t)]\{1 - E[\underline{X1}(t)]\}$$

and

$$\dot{E}[\underline{y2}^+(t)] = \dot{E}[\underline{X3}^+(t)]E[\underline{X4}(t)] + \dot{E}[\underline{X4}^+(t)]E[\underline{X3}(t)] \tag{2.104}$$

$$\dot{E}[\underline{y2}^-(t)] = \dot{E}[\underline{X3}^-(t)]E[\underline{X4}(t)] + \dot{E}[\underline{X4}^-(t)]E[\underline{X3}(t)] \quad .$$

Carrying out the convolutions in Eq.(2.90), $\dot{E}[\underline{Y1}^+(t)]$, $\dot{E}[\underline{Y1}^-(t)]$, $\dot{E}[\underline{Y2}^+(t)]$, and $\dot{E}[\underline{Y2}^-(t)]$ are obtained. Additional upper case letters are added to the subscripts of the propagation delays to denote the gate type: N-for NOR gate, A-for AND gate, and O-for OR gate. Thus,

$$\dot{E}[\underline{Y1}^+(t)] = \dot{E}[\underline{y1}^+(t)]*f_{\underline{\tau rN}}(t) \;;\; \dot{E}[\underline{Y1}^-(t) = \dot{E}[\underline{y1}^-(t)]*f_{\underline{\tau fN}}(t); \tag{2.105}$$

$$\dot{E}[\underline{Y2}^+(t)] = \dot{E}[\underline{y2}^+(t)]*f_{\underline{\tau rA}}(t) \;;\; \dot{E}[\underline{Y2}^-(t)] = \dot{E}[\underline{y2}^-(t)]*f_{\underline{\tau fA}}(t). \tag{2.106}$$

Application of Eq.(2.85) yields

$$E[\underline{Y1}(t)] = E[\underline{Y1}(0)] + \int_0^t \{\dot{E}[\underline{Y1}^+(\theta)] - \dot{E}[\underline{Y1}^-(\theta)]\}\, d\theta$$

$$E[\underline{Y2}(t)] = E[\underline{Y2}(0)] + \int_0^t \{\dot{E}[\underline{Y2}^+(\theta)] - \dot{E}[Y2^-(\theta)]\}\, d\theta. \tag{2.107}$$

Now, for independent inputs $\underline{X1}(t)$, $\underline{X2}(t)$, $\underline{X3}(t)$, and $\underline{X4}(t)$ and independent delays $\underline{\tau}_{rN}$, $\underline{\tau}_{fN}$, $\underline{\tau}_{rA}$ and $\underline{\tau}_{fA}$, observe that the two signals $\underline{Y1}(t)$ and $\underline{Y2}(t)$ are statistically independent. Therefore, the same procedure can be repeated to obtain the network output. Specifically,

$$\dot{E}[\underline{z}^+(t)] = \dot{E}[\underline{Y1}^+(t)]\{1 - E[\underline{Y2}(t)]\} + \dot{E}[\underline{Y2}^+(t)]\{1 - E[\underline{Y1}(t)]\}$$

$$\dot{E}[\underline{z}^-(t)] = \dot{E}[\underline{Y1}^-(t)]\{1 - E[\underline{Y2}(t)]\} + \dot{E}[\underline{Y2}^-(t)]\{1 - E[\underline{Y1}(t)]\} \tag{2.108}$$

and

$$E[\underline{Z}(t)] = E[\underline{Z}(0)] + \int_0^t \{\dot{E}[\underline{z}^+(\theta)]*f_{\underline{\tau}r0}(\theta) - \dot{E}[\underline{z}^-(\theta)]*f_{\underline{\tau}f0}(\theta)\}\, d\theta. \tag{2.109}$$

Eq. (2.109) is the required result. ∇

Examples 2.8 and 2.9 demonstrate the simplicity of the systematic method for evaluation of the output expected value of a tree-like network. The method becomes considerably more complicated when networks with reconvergent fanouts are considered. This is discussed in the next chapter.

3. ANALYSIS OF NETWORKS WITH RECONVERGENT FANOUTS

3.1 Introduction

As described in Chapter 1, a combinational network is said to contain reconvergent fanout if more than one path exists from an input to an output. (i.e., a network has reconvergent fanout when two or more branches originate from the same point and merge back together at some other point of the network). Example 3.1 depicts a network containing reconvergent fanout.

Example 3.1 Observe the combinational network shown in Fig. 3.1.

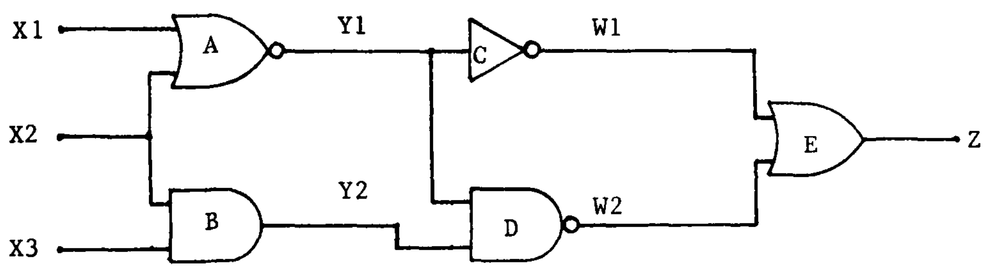


Fig. 3.1 A network with reconvergent fanout.

Note that the signal path for X1 splits at Y1 and reconverges at gate E (i.e., two paths exist from X1 to the output Z). Also,the signal path for X2 splits between gates A and B and reconverges both at gate D and gate E (i.e., three paths exist from X2 to the output Z). Observe that X3 has only one path to the output. ∇

To demonstrate the difficulties that arise in the analysis of networks containing reconvergent fanout consider the following simple example.

Example 3.2. A simple combinational network is shown in Fig. 3.2. The inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ are assumed to be statistically independent. Observe that the signal $\underline{X2}(t)$ has two paths to the output. Thus, by definition, this network contains reconvergent fanout. For simplicity,

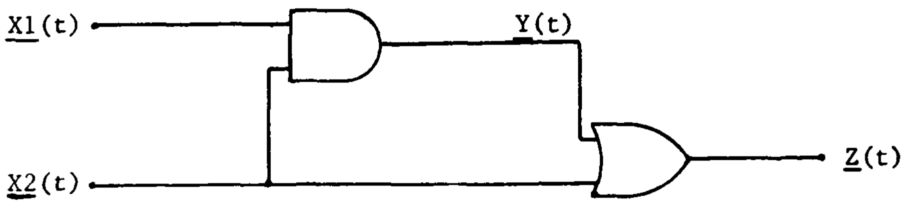


Fig. 3.2 Circuit for Ex. 3.2

suppose all the gates in Fig. 3.2 include pure delays (i.e $\underline{\tau}_r = \underline{\tau}_f = \underline{\tau}$). Following the notation of Ch.2, denote the pure delay of the AND gate by $\underline{\tau}_A$ and the pure delay of the OR gate by $\underline{\tau}_O$. The switching expression for the output is given by:

$$\underline{Z}(t) = \underline{X2}(t-\underline{\tau}_O) \vee \underline{X1}(t-\underline{\tau}_A-\underline{\tau}_O)\cdot\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_O). \quad (3.1)$$

To obtain the sum of nonoverlapping implicants form note that the random variable $\underline{X2}(t-\underline{\tau}_O)$ differs from the random variable $\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_O)$ since they are taken at different time instants. As a result, $\underline{X2}(t-\underline{\tau}_O)$ and $\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_O)$ must be treated as two different switching variables. Therefore, the sum of nonoverlapping implicants is given by

$$\underline{Z}(t) = X2(t-\underline{\tau}_O) \vee \underline{X1}(t-\underline{\tau}_A-\underline{\tau}_O)\cdot\underline{X2}(t-\tau_A-\tau_O)\cdot\underline{X2}'(t-\underline{\tau}_O) \quad . \quad (3.2)$$

Application of the probability expression theorem to Eq.(3.2), using independence of $\underline{X1}(t)$ and $\underline{X2}(t)$, yields:

$$E[\underline{Z}(t)] = E[\underline{X2}(t - \underline{\tau}_0)] + E[\underline{X1}(t-\underline{\tau}_A-\underline{\tau}_0)]E[\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_0)\underline{X2}'(t-\underline{\tau}_0)]. \quad (3.3)$$

Note that the second term of the right side of Eq.(3.3) includes the expectation $E[\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_0)\underline{X2}'(t-\underline{\tau}_0)]$, called the second order joint moment of $\underline{X2}(t)$ and $\underline{X2}'(t)$. The above second order joint moment can be further simplified into $E[\underline{X2}(t-\underline{\tau}_A-\tau_0)]-E[\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_0)\underline{X2}(t-\underline{\tau}_0)]$. The expectation $E[\underline{X2}(t-\underline{\tau}_A-\underline{\tau}_0)\underline{X2}(t-\underline{\tau}_0)]$ is referred to as a second order moment of $\underline{X2}(t)$. ∇

In general, the ℓ^{th} order moment of $\underline{Xk}(t)$ is defined to be [18,p.297]

$$R_{\underline{Xk}\ldots\underline{Xk}}(t_1,\ldots,t_\ell) = E[\underline{Xk}(t_1)\ldots\underline{Xk}(t_\ell)] , \quad (3.4)$$

while by definition, the ℓ^{th} order joint moment of the m different processes $\underline{X1}(t),\ldots,\underline{Xm}(t)$ is

$$R_{\underline{X1}\ldots\underline{X1}\ \underline{X2}\ldots\underline{Xm}}(t_1,\ldots,t_{\ell 1}, t_{\ell 1+1},\ldots,t_{\ell 2},\ldots,t_\ell)$$

$$= E[\underline{X1}(t_1)\ldots\underline{X1}(t_{\ell 1})\underline{X2}(t_{\ell 1+1})\ldots\underline{X2}(t_{\ell 2})\ldots\underline{Xm}(t_\ell)], 1 \le \ell_1 < \ell_2 < \ldots < \ell_m = \ell.$$

(3.5)

In Example 3.2, it was observed that due to the different delays, the two paths between $\underline{X2}$ and the output resulted in an expression for the expected value of the output which includes a term involving the second

order moment of $\underline{X2}$. The order of reconvergence for a signal $\underline{yk}$ is defined to be the number of paths having different delays from the node $\underline{yk}$ to the output node $\underline{Z}$. It is now shown that the order for moments of $\underline{yk}$ required in $E[\underline{Z}]$ may equal the order of reconvergence for $\underline{yk}$.

Assume the last gate preceding the output node $\underline{Z}$ is an AND gate containing m+q-1 delayed inputs as shown in Fig. 3.3. The (m-1) inputs $\underline{Yi}(t-\underline{\tau}_{i1})$, i=2,...,m, have only one path from the node $\underline{Yi}$ to the output node $\underline{Z}$, where the delay associated with the path from node $\underline{Yi}$ to the input of the AND gate is denoted by $\underline{\tau}_{i1}$. The signal at node $\underline{Y1}$, on the other

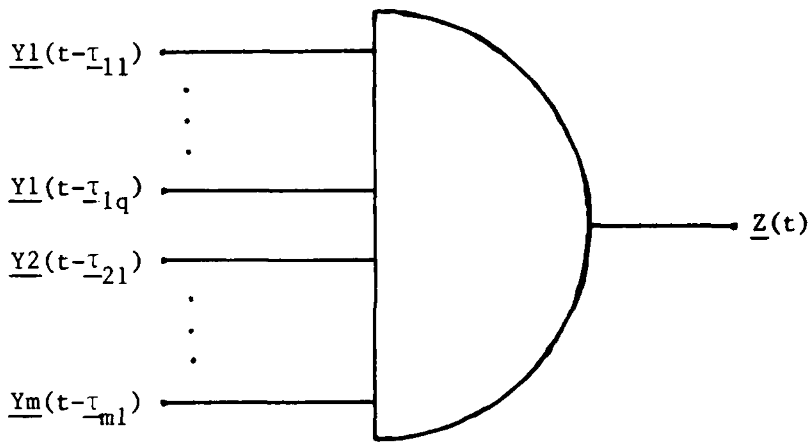


Fig. 3.3. AND gate containing m+q-1 delayed inputs.

hand, has reconvergence of the order q (i.e., there are q paths with different delays from the node $\underline{Y1}$ to the output node $\underline{Z}$). The delays from the node $\underline{Y1}$ to the corresponding inputs of the AND gate are denoted by $\underline{\tau}_{11},\ldots,\underline{\tau}_{1q}$. All signals $\underline{Y1}(t),\ldots,\underline{Ym}(t)$ are assumed to be statistically independent. The output $\underline{Z}(t)$ is given by

$$\underline{Z}(t) = \underline{Y1}(t-\underline{\tau}_{11})\wedge\ldots\wedge \underline{Y1}(t-\underline{\tau}_{1q})\wedge Y2(t-\underline{\tau}_{21})\wedge\ldots\wedge Ym(t-\underline{\tau}_{m1}) \tag{3.6}$$

Observe that Eq.(3.6) is in the form of a sum of nonoverlapping implicants where the sum contains a single term. Applying the probability expression theorem,

$$E[\underline{Z}(t)] = E[\underline{Y1}(t-\underline{\tau}_{11})\ldots\underline{Y1}(t-\underline{\tau}_{1q})\underline{Y2}(t-\underline{\tau}_{21})\ldots\underline{Ym}(t-\underline{\tau}_{m1})]. \qquad (3.7)$$

Utilizing the statistical independence among $\underline{Y1}(t),\ldots,\underline{Ym}(t)$, Eq.(3.7) is written as:

$$E[\underline{Z}(t)] = E[\underline{Y1}(t-\underline{\tau}_{11})\ldots\underline{Y1}(t-\underline{\tau}_{1q})]\; E[\underline{Y2}(t-\underline{\tau}_{2i})]\ldots E[\underline{Ym}(t-\underline{\tau}_{m1})]. \qquad (3.8)$$

It is observed that the output expected value, E[Z(t)], includes the q^{th} order moment of $\underline{Y1}(t)$. But q is also the order of reconvergence of the signal $\underline{Y1}(t)$. Thus, in this case, the order required for the moments of $\underline{Y1}(t)$ in the expected value of the output signal $\underline{Z}(t)$ equals the order of reconvergence of $\underline{Y1}(t)$.

However, the order required for the moment of $\underline{Yk}(t)$ does not necessarily equal the corresponding order of reconvergence. It may be lower if some of the paths are maintained at a constant value. This is illustrated by the following example.

Example 3.3. Consider the network shown in Fig. 3.4.

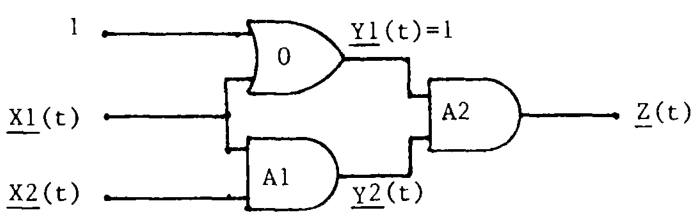


Fig. 3.4. Network with one input held at 1.

Observe that there are two different paths between the input node $\underline{X1}$ and the output node $\underline{Z}$. However, since one of the inputs to the OR gate is maintained at the value 1, the output $\underline{Y1}(t)$ is also held at the value 1. Therefore, the output $\underline{Z}(t)$ is affected by only one of the paths involving $\underline{X1}(t)$. In particular, denoting the delays of gates A1 and A2 by $\underline{\tau}_{A1}$ and $\underline{\tau}_{A2}$, respectively, the output $\underline{Z}(t)$ is given by

$$\underline{Z}(t) = \underline{X1}(t-\underline{\tau}_{A1}-\underline{\tau}_{A2})\ \underline{X2}(t-\underline{\tau}_{A1}-\underline{\tau}_{A2}). \tag{3.9}$$

Assuming the inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ are statistically independent and using the conditional expectation theorem, the expected value of $\underline{Z}(t)$ is given by

$$E[\underline{Z}(t)] = \int_0^\infty \int_0^\infty E_{\underline{X1}(t)}[\underline{X1}(t-\tau_{A1}-\tau_{A2})] \tag{3.10}$$

$$E_{\underline{X2}(t)}[\underline{X2}(t-\tau_{A1}-\tau_{A2})]\ f_{\underline{\tau}A1}(\tau_{A1}) f_{\underline{\tau}A2}(\tau_{A2}) d\tau_{A1}\ d\tau_{A2}.$$

Observe that only first order moments arise in Eq.(3.10) even though the order of reconvergence for $\underline{X1}(t)$ is two. ∇

When discriminating delay elements in the network are considered, the signals must be decomposed into their counting signals. As a result, four different signals may be encountered for every signal $\underline{Yk}(t)$, in evaluation of the derivatives $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$. These are $\underline{Yk}^{+}(t)$, $\underline{Yk}^{-}(t)$, $\dot{\underline{Yk}}^{+}(t)$, and $\dot{\underline{Yk}}^{-}(t)$. If the order of reconvergence for $\underline{Yk}$ is q, then the analysis for $E[\underline{Z}(t)]$ may require joint moments of orders up to q involving $\underline{Yk}^{+}(t)$, $\underline{Yk}^{-}(t)$, $\dot{\underline{Yk}}^{+}(t)$ and $\dot{\underline{Yk}}^{-}(t)$. These terms can get quite large in number since the number of different ways to place

4 objects in γ cells with repetition is given by [22,p.909]

$$\binom{\gamma + 3}{\gamma} = \frac{(\gamma+3)!}{\gamma!\ 3!} \tag{3.11}$$

Example 3.4. For γ=2 there are $\frac{5!}{2!\ 3!}$ = 10 possible second order joint moments (correlation functions) involving the counting signals of $\underline{Yk}(t)$ and their derivatives.

Specifically, the ten possible correlations function for $\underline{Yk}(t)$ are listed below:

$$R_{\underline{Yk}^+\underline{Yk}^+}(t_1,t_2),\ R_{\underline{Yk}^+\underline{Yk}^-}(t_1,t_2),\ R_{\underline{Yk}^+\underline{\dot{Y}k}^+}(t_1,t_2),\ R_{\underline{Yk}^+\underline{\dot{Y}k}^-}(t_1,t_2),\ R_{\underline{Yk}^-\underline{Yk}^-}(t_1,t_2)$$

$$R_{\underline{Yk}^-\underline{\dot{Y}k}^+}(t_1,t_2), R_{\underline{Yk}^-\underline{\dot{Y}k}^-}(t_1,t_2), R_{\underline{\dot{Y}k}^+\underline{\dot{Y}k}^+}(t_1,t_2),\ R_{\underline{\dot{Y}k}^+\underline{\dot{Y}k}^-}(t_1,t_2),\ R_{\underline{\dot{Y}k}^-\underline{\dot{Y}k}^-}(t_1,t_2). \tag{3.12}$$

∇

Methods for reducing the number of needed higher order moments are discussed in Sec. 3.3., where discriminating delay elements are considered. As in Chapter 2, the simpler case of pure delay is discussed first.

3.2. Pure Delay

A typical logic block with a pure delay element, i.e. $\tau_r = \tau_f = \tau$, is illustrated in Fig. 3.5 (same as Fig.2.3).

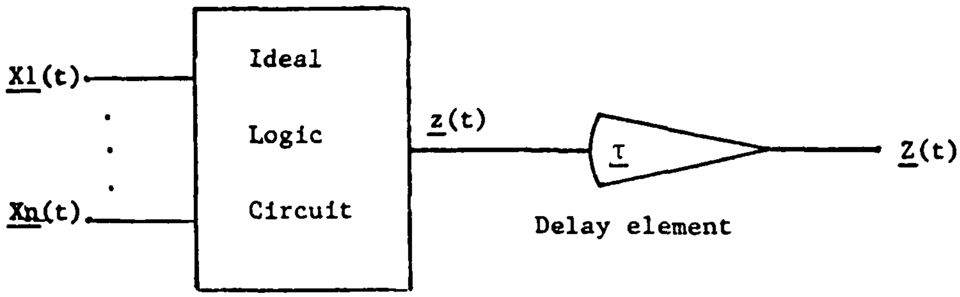


Fig. 3.5 A typical logic block with a pure delay element.

Again, for ease of discussion, a single output $\underline{Z}(t)$ is considered. The inputs $\underline{X1}(t),\ldots,\underline{Xn}(t)$ are assumed to be stochastic 0,1 binary processes which are not necessarily statistically independent.

When the output signal, $\underline{Z}(t)$ is used as an input to a second network having reconvergent fanout, the higher order moments of $\underline{Z}(t)$ are required for evaluation of the output expected value of the second network. Therefore, in this section it is shown how to obtain higher order statistics of $\underline{Z}(t)$ in terms of the higher order statistics of the inputs $\underline{X1}(t),\ldots,\underline{Xn}(t)$ and the p.d.f. $f_{\underline{\tau}}(\tau)$.

For the pure delay case the sample functions of $\underline{Z}(t)$ are delayed replicas of the sample functions of $\underline{z}(t)$. Specifically,

$$\underline{Z}(t) = \underline{z}(t-\underline{\tau}). \tag{3.13}$$

Therefore, the ℓ^{th} order moment of $\underline{Z}(t)$ is given by

$$E[\underline{Z}(t_1)\ldots\underline{Z}(t_\ell)] = E[\underline{z}(t_1-\underline{\tau})\ldots\underline{z}(t_\ell-\underline{\tau})]. \tag{3.14}$$

The expectation in Eq.(3.14) is taken with respect to both $\underline{z}(t)$ and $\underline{\tau}$. Employing the conditional expectation theorem [18, p.208] and using the notation of Sec. 2.2, one obtains

$$R_{\underline{Z}\ldots\underline{Z}}(t_1,t_2,\ldots,t_\ell)$$

$$= E[\underline{Z}(t_1)\ldots\underline{Z}(t_\ell)] = E_{\underline{\tau}}[E_{\underline{z}}[\underline{z}(t_1-\underline{\tau})\ldots\underline{z}(t_\ell - \underline{\tau})|\underline{\tau} = \tau\ ]]$$

$$= \int_0^\infty E_{\underline{z}}[\underline{z}(t_1 - \underline{\tau})\ldots\underline{z}(t_\ell - \underline{\tau})|\underline{\tau} = \tau]\ f_{\underline{\tau}}(\tau)\ d\tau$$

$$= \int_0^\infty E[\underline{z}(t_1 -\tau)\ \ldots\ \underline{z}(t_\ell - \tau)]f_{\underline{\tau}}(\tau)d\tau. \qquad (3.15$$

The following simple example illustrates this evaluation.

Example 3.5 Suppose z(t) is deterministic and consists of a single pulse between 0 and 2. Let the delay $\underline{\tau}$ be uniformly distributed between 0.5 and 1. From Eq. (3.15), the output autocorrelation is given by

$$R_{\underline{ZZ}}(t_1,t_2) = \int_0^\infty E[z(t_1 -\tau)z(t_2-\tau)]f_{\underline{\tau}}(\tau)\ d\tau\ . \qquad (3.16$$

Because z(t) is deterministic and τ is the variable of integration

$$E[z(t_1-\tau)z(t_2- \tau)] = z(t_1 - \tau)z(t_2 - \tau)\quad . \qquad (3-17$$

Hence,

$$R_{\underline{ZZ}}(t_1,t_2)= \int_0^\infty z(t_1 - \tau)z(t_2 - \tau)f_{\underline{\tau}}(\tau)d\tau$$

$$= 2\int_{0.5}^{1.0} z(t_1 - \tau)z(t_2 - \tau)d\tau. \qquad (3-18$$

$z(t)$, $f_{\underline{\tau}}(\tau)$, the input autocorrelation function and the output autocorrelation function are sketched in Fig. 3.6.

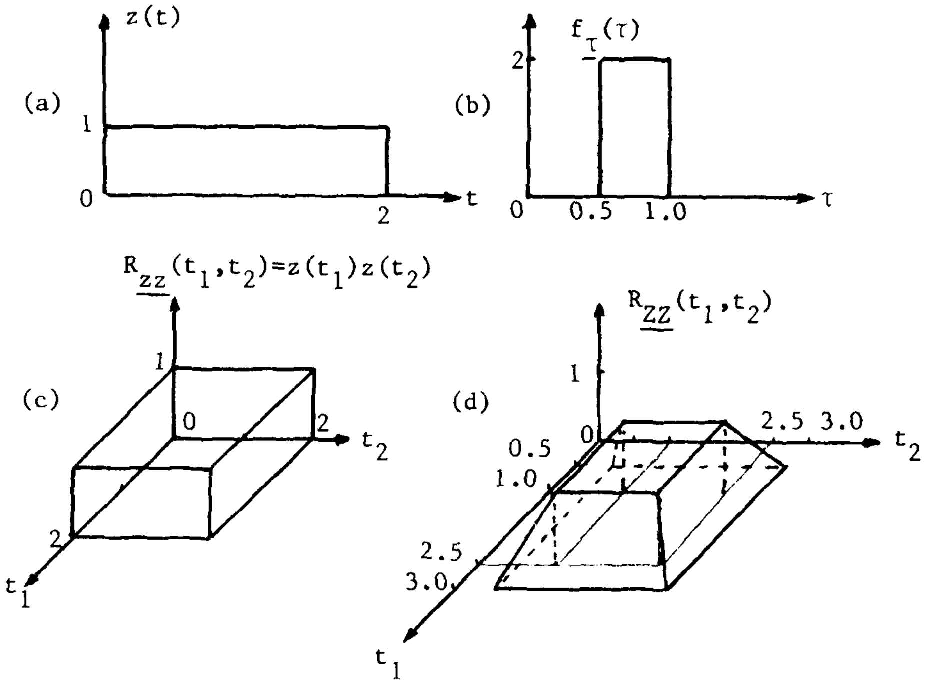


Fig. 3.6. (a) signal $z(t)$ (b) p.d.f. $f_{\underline{\tau}}(\tau)$ (c) Input autocorrelation function and (d) output autocorrelation function.

The output autocorrelation function is seen to be a shifted and distorted version of the input autocorrelation function due to the random delay. ∇

In general, knowledge of $E[\underline{z}(t_1 - \tau)\ldots\underline{z}(t_\ell - \tau)]$ is necessary in order to evaluate Eq. (3.15). For this purpose, it is convenient to derive $E[\underline{z}(t_1)\ldots\underline{z}(t_\ell)]$ in terms of the higher order moments of the input switching variable $\underline{X1}(t),\ldots,\underline{Xn}(t)$. It is now shown that the arithmetic expression theorem from Ch. 2 is also applicable here.

Consider the AND gate in Fig. 3.7 whose inputs are $\underline{z}(t_1),\ldots,\underline{z}(t_\ell)$.

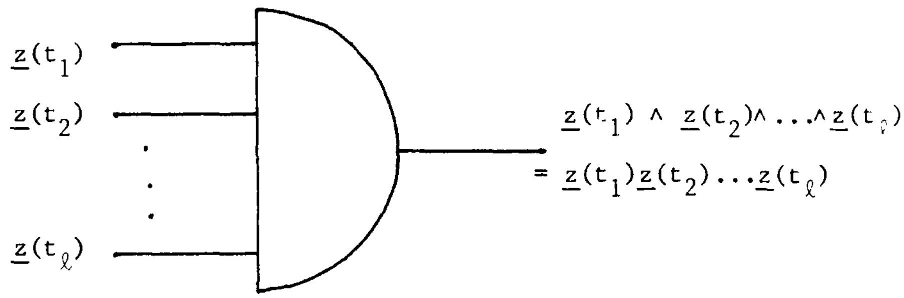


Fig. 3.7 AND gate with inputs $\underline{z}(t_1),\ldots,\underline{z}(t_\ell)$.

Let each switching function $\underline{z}(t_k)$, $k = 1,\ldots,\ell$, be expressed as a sum of nonoverlapping implicants. Observe that performing an AND operation between a set of nonoverlapping implicants and other switching variables generate another set of nonoverlapping implicants. It follows that application of the distributive law to the output $\underline{z}(t_1) \wedge \underline{z}(t_2) \wedge \ldots \wedge \underline{z}(t_\ell)$ yields a sum of nonoverlapping implicants. Use of the arithmetic expression theorem from Ch. 2 then results in an arithmetic expression for the output. The output expected value $E[\underline{z}(t_1)\ldots\underline{z}(t_\ell)]$ is next obtained by recognizing that the expectation of a sum equals the sum of the expectations. Finally, the expression can be simplified by taking advantage of statistical independence where applicable.

A useful tool in the evaluation of $E[\underline{z}(t_1)\ldots\underline{z}(t_\ell)]$ is to recognize an equivalent approach which is to replace each switching function $\underline{z}(t_k)$, $k = 1,\ldots,\ell$, by its arithmetic expression and then apply the distributive law on the arithmetic expression.

__Example 3.6__ Determine the third order moment of $\underline{z}(t)$, $E[\underline{z}(t_1)\underline{z}(t_2)\underline{z}(t_3)]$, for $\underline{z}(t) = \underline{X1}(t) \vee \underline{X2}(t)$. The inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ are assumed to be statistically independent.

An arithmetic expression for $\underline{z}(t)$ is given by

$$\underline{z}(t) = \underline{X1}(t) + \underline{X1}'(t)\underline{X2}(t). \tag{3.19}$$

Consequently,

$$\underline{z}(t_1)\underline{z}(t_2)\underline{z}(t_3) = [\underline{X1}(t_1)+ \underline{X1}'(t_1)\underline{X2}(t_1)][\underline{X1}(t_2) + \underline{X1}'(t_2)\underline{X2}(t_2)][\underline{X1}(t_3) + \underline{X1}'(t_3)\underline{X2}(t_3)]\ . \tag{3.20}$$

Application of the distributive law results in

$$\begin{aligned}\underline{z}(t_1)\underline{z}(t_2)\underline{z}(t_3) = {} & [\underline{X1}(t_1)\underline{X1}(t_2)\underline{X1}(t_3)+ \underline{X1}(t_1)\underline{X1}'(t_2)\underline{X1}(t_3)\underline{X2}(t_2) \\ & + \underline{X1}'(t_1)\underline{X1}(t_2)\quad \underline{X1}(t_3)\ \underline{X2}(t_1)+ \underline{X1}'(t_1)\underline{X1}'(t_2)\underline{X1}(t_3)\underline{X2}(t_1)\underline{X2}(t_2) \\ & +\underline{X1}(t_1)\underline{X1}(t_2)\underline{X1}'(t_3)\underline{X2}(t_3)+ \underline{X1}(t_1)\underline{X1}'(t_2)\underline{X1}'(t_3)\underline{X2}(t_2)\underline{X2}(t_3) \\ & + \underline{X1}'(t_1)\underline{X1}(t_2)\underline{X1}'(t_3)\underline{X2}(t_1)\underline{X2}(t_3) \\ & + \underline{X1}'(t_1)\underline{X1}'(t_2)\underline{X1}'(t_3)\underline{X2}(t_1)\underline{X2}(t_2)\underline{X2}(t_3)\end{aligned} \tag{3.21}$$

Taking the expectation of Eq.(3.21) and using statistical independence when possible

$$E[\underline{z}(t_1)\underline{z}(t_2)\underline{z}(t_3)] = E[\underline{X1}(t_1)\underline{X1}(t_2)\underline{X1}(t_3)] + E[\underline{X1}(t_1)\underline{X1}'(t_2)X1(t_3)]E[\underline{X2}(t_2)]$$

$$+ E[\underline{X1}'(t_1)\underline{X1}(t_2)\underline{X1}(t_3)]E[\underline{X2}(t_1)] + E[\underline{X1}'(t_1)\underline{X1}'(t_2)\underline{X1}(t_3)]R_{\underline{X2X2}}(t_1,t_2)$$

$$+ E[\underline{X1}(t_1)\underline{X1}(t_2)\underline{X1}'(t_3)]E[\underline{X2}(t_3)] + E[\underline{X1}(t_1)\underline{X1}'(t_2)\underline{X1}'(t_3)]R_{\underline{X2X2}}(t_2,t_3)$$

$$+ E[\underline{X1}'(t_1)\underline{X1}(t_2)\underline{X1}'(t_3)]R_{\underline{X2}\ \underline{X2}}(t_1,t_3)$$

$$+ E[\underline{X1}'(t_1)\underline{X1}'(t_2)\underline{X1}'(t_3)]E[\underline{X2}(t_1)\underline{X2}(t_2)\underline{X2}(t_3)]. \tag{3.22}$$

As a final point, note that complemented variables in Eq. (3.22) could have been eliminated by using in Eq.(3.20) the property that

$$\underline{Xk}'(t) = 1 - \underline{Xk}(t). \qquad \nabla \tag{3.23}$$

In general, networks containing reconvergent fanout also contain tree-like subnetworks. The tree-like subnetworks can be analyzed using the procedures described in Sec. 2.2. The higher order moments which arise due to the reconvergences are obtained as explained in this section. The procedure is illustrated in the following network example.

Example 3.7 Consider the combinational network shown in Figure 3.8. All gates are assumed to contain pure delay elements. Also, the inputs $\underline{Xk}(t)$, k=1,2,3 are assumed to be statistically independent. Since there are two paths to the output for $\underline{X2}(t)$ and $\underline{X3}(t)$, the network contains a reconvergent fanout. It is desired to evaluate the expected value of the output $E[\underline{Z}(t)]$.

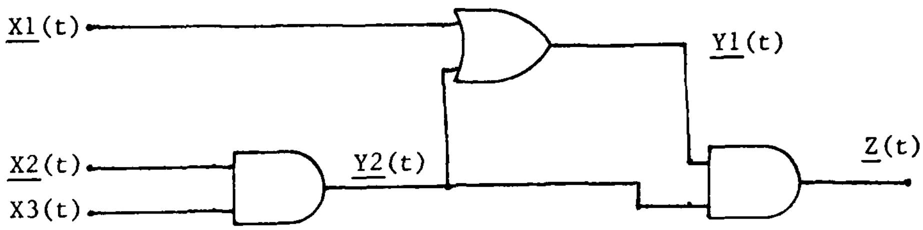


Fig. 3.8. Combinational network with a reconvergent fanout.

The model for the physical network in Fig. 3.8 is illustrated in Fig. 3.9. Observe the ideal logic output signals $\underline{y1}(t)$, $\underline{y2}(t)$, and $\underline{z}(t)$ and the pure delay elements with random delays $\underline{\tau}_A$ and $\underline{\tau}_O$.

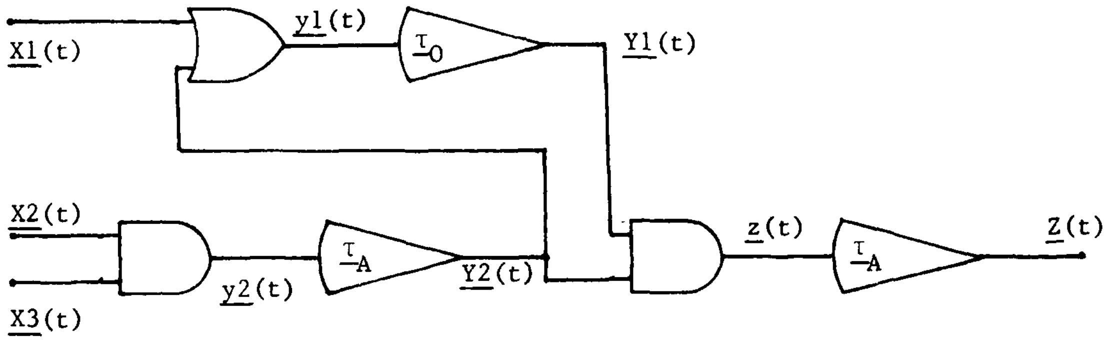


Fig. 3.9. Model for network in Fig. 3.8.

The expectation of the ideal logic output $\underline{y2}(t)$ is given by

$$E[\underline{y2}(t)] = E[\underline{X2}(t)]E[\underline{X3}(t)]. \tag{3.24}$$

Using Eq.(2.6) the expected value of the gate output $\underline{Y2}(t)$ is

$$E[\underline{Y2}(t)] = E[\underline{y2}(t)]*f_{\underline{\tau}A}(t) = \{E[\underline{X2}(t)]E[\underline{X3}(t)]\}*f_{\underline{\tau}A}(t). \tag{3.25}$$

The expected value of $\underline{Y1}(t)$ is also readily obtained. Noting that $\underline{X1}(t)$ and $\underline{Y2}(t)$ are statistically independent the expected value of $\underline{y1}(t)$ is given by

$$E[\underline{y1}(t)] = E[\underline{X1}(t)] + E[\underline{Y2}(t)]-E[\underline{X1}(t)]E[\underline{Y2}(t)]. \tag{3.26}$$

Once agin, by means of Eq.(2.6),

$$E[\underline{Y1}(t)] = E[\underline{y}1(t)]*f_{\underline{\tau}0}(t)$$

$$= E[\underline{X1}(t)]*f_{\underline{\tau}0}(t) + E[\underline{Y2}(t)]*f_{\underline{\tau}0}(t) - \{E[\underline{X1}(t)]E[\underline{Y2}(t)]\}*f_{\underline{\tau}0}(t). \tag{3.27}$$

with respect to the last AND gate, the expected value of $\underline{z}(t)$ is given by

$$E[\underline{z}(t)] = E[\underline{Y1}(t)\underline{Y2}(t)]. \tag{3.28}$$

Note that $\underline{Y1}(t)$ and $\underline{Y2}(t)$ are not statistically independent. The arithmetic expression for $\underline{Y1}(t)$ is given by

$$\underline{Y1}(t) = \underline{X1}(t-\underline{\tau}_0)+ \underline{Y2}(t -\underline{\tau}_0)-\underline{X1}(t-\underline{\tau}_0)\underline{Y2}(t-\underline{\tau}_0). \tag{3.29}$$

Substituting Eq.(3.29) into Eq.(3.28) and using the independence of $\underline{X1}(t)$ and $\underline{Y2}(t)$ one obtains

$$E[\underline{z}(t)]= E[\underline{X1}(t-\underline{\tau}_0)\underline{Y2}(t)+\underline{Y2}(t-\underline{\tau}_0)\underline{Y2}(t) - \underline{X1}(t-\underline{\tau}_0)\underline{Y2}(t-\underline{\tau}_0)\underline{Y2}(t)]$$

$$= E[\underline{X1}(t-\underline{\tau}_0)]E[\underline{Y2}(t)] + R_{\underline{Y2}\ \underline{Y2}}(t,t-\underline{\tau}_0)-E[\underline{X1}(t-\underline{\tau}_0)]R_{\underline{Y2}\ \underline{Y2}}(t,t-\tau_0). \tag{3.30}$$

Using the conditional expectation theorem [18] to average over the delay $\underline{\tau}_0$, Eq.(3.30) becomes

$$E[\underline{z}(t)] = \{E[\underline{X1}(t)]*f_{\underline{\tau}0}(t)\}E[\underline{Y2}(t)] + \int_0^\infty R_{\underline{Y2}\ \underline{Y2}}(t,t-\tau)f_{\underline{\tau}0}(\tau)d\tau$$

$$- \int_0^\infty E[\underline{X1}(t-\tau)R_{\underline{Y2}\ \underline{Y2}}(t,t-\tau)f_{\underline{\tau}0}(\tau)d\tau \ . \qquad (3.31)$$

Observe that evaluation of Eq.(3.31) requires knowledge of the autocorrelation function for $\underline{Y2}(t)$. By definition the autocorrelation function of the ideal logic output $\underline{y2}(t)$ is

$$R_{\underline{y2}\ \underline{y2}}(t,t-\tau) = E[\underline{y2}(t)\underline{y2}(t-\tau)] = E[\underline{X2}(t)\underline{X3}(t)\underline{X2}(t-\tau)\underline{X3}(t-\tau)]$$

$$= R_{\underline{X2}\ \underline{X2}}(t,t-\tau)R_{\underline{X3}\ \underline{X3}}(t,t-\tau) \ . \qquad (3.32)$$

Making use of Eq.(3.15), the autocorrelation function of $\underline{Y2}(t)$ is

$$R_{\underline{Y2}\ \underline{Y2}}(t,t-\tau) = \int_0^\infty R_{\underline{y2}\ \underline{y2}}(t-\alpha,t-\tau-\alpha)f_{\underline{\tau}A}(\alpha)d\alpha$$

$$= \int_0^\infty R_{\underline{X2}\ \underline{X2}}(t-\alpha,t-\tau-\alpha)R_{\underline{X3}\ \underline{X3}}(t-\alpha,t-\tau-\alpha)f_{\underline{\tau}A}(\alpha)d\alpha . \qquad (3.33)$$

Substituting into Eq.(3.31), Eqs.(3.25) and (3.33) for $E[\underline{Y2}(t)]$ and $R_{\underline{Y2}\ \underline{Y2}}(t,t-\tau)$, respectively, an expression for $E[\underline{z}(t)]$ is obtained in terms of the inputs $\underline{X1}(t)$, $\underline{X2}(t)$, and $\underline{X3}(t)$.

Finally, from Eq.(2.6) $E[\underline{Z}(t)]$ is given by

$$E[\underline{Z}(t)] = E[\underline{z}(t)]*f_{\underline{\tau}A}(t) \ . \qquad (3.34)$$

Note that in Eq.(3.34) first and second order moments for the reconverging signals $\underline{X2}(t)$ and $\underline{X3}(t)$ are needed while only the mean of $\underline{X1}(t)$ is required. ∇

In summary, the procedure for obtaining the ouput expected value of a combinational network with pure delay elements is performed by progressing from the inputs to the output, logic block by logic block. For the tree-like portion, the expectation of a block output is evaluated in terms of the expectations of the block inputs and the p.d.f. of the corresponding delay. When a reconvergent fanout is encountered, the ideal logic output signal must be expressed explicitly in terms of the reconverging signals using the arithmetic expression theorem. Taking the expected value of the arithmetic expression results in higher order moments as illustrated in Eq.(3.30). The higher order moments can always be expressed in terms of the higher order moments of the primary inputs, as illustrated in Eq.(3.33). As was done in Eq.(3.31), it is often useful to employ the conditional expectation theorem. Finally, the output expected value, $E[\underline{Z}(t)]$, is obtained by convolving $E[\underline{z}(t)]$ with the p.d.f. of the output delay, as in Eq.(3.34).

3.3. Discriminating Delay

The model of a typical logic block with a single output and a discriminating delay($\tau_r \neq \tau_f$) is illustrated in Fig. 3.10 (same as Fig. 2.5).

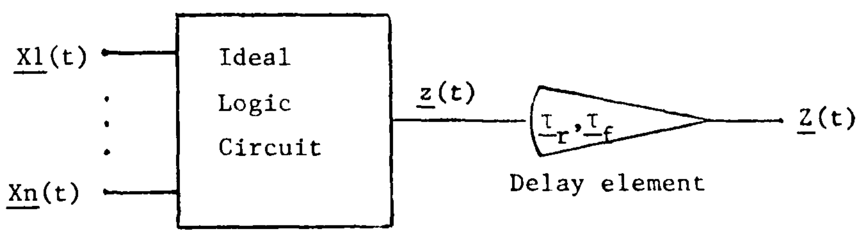


Fig. 3.10. The model for a typical logic block with $\tau_r \neq \tau_f$.

It was pointed out in Ch.1 that in such a case the signal $\underline{Z}(t)$ must be decomposed into the difference of two counting signals as given by

$$\underline{Z}(t) = \underline{Z}^{+}(t) - \underline{Z}^{-}(t). \tag{3.35}$$

However, a direct derivation for $\underline{Z}^{+}(t)$ and $\underline{Z}^{-}(t)$ in terms of the input signals $\underline{X1}(t),\ldots,\underline{Xn}(t)$ and/or their counting signals is not known. As in Ch.2, the difficulty is overcome by introducing the time derivatives $\dot{\underline{Z}}^{+}(t)$ and $\dot{\underline{Z}}^{-}(t)$. When reconvergence is encountered, the higher order joint moments of $\underline{Z}^{+}(t)$, $\underline{Z}^{-}(t)$, $\dot{\underline{Z}}^{+}(t)$ and $\dot{\underline{Z}}^{-}(t)$ are required. Their derivation is presented in this section. Because a very large number of such moments may arise, an efficient procedure for performing the network analysis is also developed.

Suppose the output $\underline{Z}(t)$ of the logic block shown in Fig. 3.10 is used as an input to a second network containing reconvergent fanout. Analysis of the second network requires higher order joint moments of $\underline{Z}^{+}(t)$, $\underline{Z}^{-}(t)$, $\dot{\underline{Z}}^{+}(t)$ and $\dot{\underline{Z}}^{-}(t)$. Denote the ℓ^{th} order joint moment by

$$E[\underline{Z}^{+}(t_1)\ldots\underline{Z}^{+}(t_{\ell 1})\underline{Z}^{-}(t_{\ell 1+1})\ldots\underline{Z}^{-}(t_{\ell 2})\dot{\underline{Z}}^{+}(t_{\ell 2+1})\ldots\dot{\underline{Z}}^{+}(t_{\ell 3})\dot{\underline{Z}}^{-}(t_{\ell 3+1})\ldots\dot{\underline{Z}}^{-}(t_{\ell})];$$

$$0 < \ell 1 \leq \ell 2 \leq \ell 3 < \ell.$$

Recall that $\underline{Z}^{+}(t)$ is the rise transition counting process and $\dot{Z}^{+}(t)$ is its derivative. Also $\underline{Z}^{-}(t)$ is the fall transition counting process and $\dot{\underline{Z}}^{-}(t)$ is its derivative. As discussed in Ch. 2, these processes can be expressed in terms of the delay element input processes

$\underline{z}^+(t)$, $\underline{z}^-(t)$, $\dot{z}^+(t)$, and $\dot{\underline{z}}^-(t)$ as follows:

$$\underline{Z}^+(t) = \underline{z}^+(t-\underline{\tau}_r),\ \dot{\underline{Z}}^+(t) = \dot{\underline{z}}^+(t-\underline{\tau}_r)$$

$$\underline{Z}^-(t) = \underline{z}^-(t-\underline{\tau}_f),\ \dot{\underline{Z}}^-(t) = \dot{z}^-(t-\underline{\tau}_f). \tag{3.36}$$

Substituting Eqs.(3.36) into the ℓ^{th} order joint moment yields

$$E[\underline{Z}^+(t_1)\ldots\underline{Z}^+(t_{\ell 1})\underline{Z}^-(t_{\ell 1+1})\ldots\underline{Z}^-(t_{\ell 2})\dot{\underline{Z}}^+(t_{\ell 2+1})\ldots\dot{\underline{Z}}^+(t_{\ell 3})\dot{\underline{Z}}^-(t_{\ell 3+1})\ldots\dot{\underline{Z}}^-(t_\ell)]$$

$$= E[\underline{z}^+(t_1-\underline{\tau}_r)\ldots\underline{z}^+(t_{\ell 1}-\underline{\tau}_r)\underline{z}^-(t_{\ell 1+1}-\underline{\tau}_f)\ldots\underline{z}^-(t_{\ell 2}-\underline{\tau}_f)\dot{\underline{z}}^+(t_{\ell 2+1}-\underline{\tau}_r)$$

$$\ldots\dot{\underline{z}}^+(t_{\ell 3}-\underline{\tau}_r)\dot{\underline{z}}^-(t_{\ell 3+1}-\underline{\tau}_f)\ldots\dot{\underline{z}}^-(t_\ell-\underline{\tau}_f)]. \tag{3.37}$$

Application of the conditional expectation theorem to the right side of Eq.(3.37) results in

$$E[\underline{Z}^+(t_1)\ldots\underline{Z}^+(t_{\ell 1})\underline{Z}^-(t_{\ell 1+1})\ldots\underline{Z}^-(t_{\ell 2})\dot{\underline{Z}}^+(t_{\ell 2+1})\ldots\dot{\underline{Z}}^+(t_{\ell 3})\dot{\underline{Z}}^-(t_{\ell 3+1})\ldots\dot{\underline{Z}}^-(t_\ell)]$$

$$= \int_0^\infty\int_0^\infty E[\underline{z}^+(t_1-\tau_r)\ldots\underline{z}^+(t_{\ell 1}-\tau_r)\underline{z}^-(t_{\ell 1+1}-\tau_f)\ldots\underline{z}^-(t_{\ell 2}-\tau_f)\dot{\underline{z}}^+(t_{\ell 2+1}-\tau_r)\ldots$$

$$\ldots\ \dot{\underline{z}}^+(t_{\ell 3}-\tau_r)\dot{\underline{z}}^-(t_{\ell 3+1}-\tau_f)\ldots\dot{\underline{z}}^-(t_\ell-\tau_f)]f_{\underline{\tau}_r\underline{\tau}_f}(\tau_r,\tau_f)d\tau_r d\tau_f\ . \tag{3.38}$$

In general, whenever the binary signal $\underline{Z}(t)$ or its derivative $\dot{\underline{Z}}(t)$ are directly involved in an expression, they can alwa;s be decomposed into

$\underline{Z}(t) = \underline{Z}^{+}(t) - \underline{Z}^{-}(t)$ or $\dot{\underline{Z}}(t) = \dot{\underline{Z}}^{+}(t) - \dot{\underline{Z}}^{-}(t)$, respectively.

Chapter 2 discusses how to obtain the derivatives of the counting processes at the output of an ideal logic circuit. Therefore, terms in Eq.(3.38) involving $\underline{z}^{+}(t)$ and $\underline{z}^{-}(t)$ can be obtained by integrating with respect to time the derivatives $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$. The integrals are assumed to exist in a mean square sense. (This simply requires that the respective autocorrelation functions be integrable over the region of integration.) Therefore, selecting t = 0 to be the time reference, $\underline{z}^{+}(t_i - \underline{\tau}_r)$ and $\underline{z}^{-}(t_j - \underline{\tau}_f)$ may be expressed as

$$\underline{z}^{+}(t_i - \underline{\tau}_r) = \underline{z}^{+}(0) + \int_0^{t_i - \underline{\tau}_r} \dot{\underline{z}}^{+}(\theta_i)d\theta_i$$

and

$$\underline{z}^{-}(t_j - \underline{\tau}_f) = \underline{z}^{-}(0) + \int_0^{t_j - \underline{\tau}_f} \dot{\underline{z}}^{-}(\theta_j)d\theta_j \qquad (3.39)$$

Without loss of generality, it is convenient to define

$$\underline{z}^{-}(0) = 0. \qquad (3.40)$$

This implies that $\underline{z}^{-}(\underline{d}_1) = 1$ and that

$$\underline{z}^{+}(0) = \underline{z}(0) \qquad (3.41)$$

is a 0,1 binary random variable. Substituting Eqs(3.39) - (3.41) into

Eq.(3.38) results in

$$E[\underline{Z}^+(t_1)\ldots\underline{Z}^+(t_{\ell 1})\underline{Z}^-(t_{\ell 1+1})\ldots\underline{Z}^-(t_{\ell 2})\dot{\underline{Z}}^+(t_{\ell 2+1})\ldots\dot{\underline{Z}}^+(t_{\ell 3})\dot{\underline{Z}}^-(t_{\ell 3+1})\ldots\dot{\underline{Z}}^-(t_\ell)]$$

$$= \int_0^\infty \int_0^\infty E[\{\underline{z}(0)+\int_0^{t_1-\tau_r}\dot{\underline{z}}^+(\theta_1)d\theta_1\}\ldots\{\underline{z}(0)+\int_0^{t_{\ell 1}-\tau_r}\dot{\underline{z}}^+(\theta_{\ell 1})d\theta_{\ell 1}\}\int_0^{t_{\ell 1+1}-\tau_f}\dot{\underline{z}}^-(\theta_{\ell 1+1})d\theta_{\ell 1+1}$$

$$\ldots\int_0^{t_{\ell 2}-\tau_f}\dot{\underline{z}}^-(\theta_{\ell 2})d\theta_{\ell 2}\,\dot{\underline{z}}^+(t_{\ell 2+1}-\tau_r)\ldots\dot{\underline{z}}^+(t_{\ell 3}-\tau_r)\dot{\underline{z}}^-(t_{\ell 3+1}-\tau_f)\ldots$$

$$\ldots\dot{\underline{z}}^-(t_\ell-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f. \tag{3.42}$$

After carrying out the products in Eq.(3.42), the final result can be simplified further by interchanging the order of integration and expectation. This latter step is permissable due to the linearity of the integration operation.

Example 3.8 Evaluate

$$E[\underline{Z}^+(t_1)\underline{Z}^+(t_2)\underline{Z}^-(t_3)\dot{\underline{Z}}^+(t_4)\dot{\underline{Z}}^-(t_5)]$$

$$= \int_0^\infty \int_0^\infty E[\{\underline{z}(0)+\int_0^{t_1-\tau_r}\dot{\underline{z}}^+(\theta_1)d\theta_1\}\{\underline{z}(0)+\int_0^{t_2-\tau_r}\dot{\underline{z}}^+(\theta_2)d\theta_2\}\int_0^{t_3-\tau_f}\dot{\underline{z}}^-(\theta_3)d\theta_3$$

$$\dot{\underline{z}}^+(t_4-\tau_r)\dot{\underline{z}}^-(t_5-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f. \tag{3.43}$$

Carrying out the products in Eq.(3.43) and using the idempotency property of the binary variable $\underline{z}(0)$ (i.e., $\underline{z}(0)\underline{z}(0) = \underline{z}(0)$) yields

$$E[\underline{Z}^{+}(t_1)\underline{Z}^{+}(t_2)\underline{Z}^{-}(t_3)\dot{\underline{Z}}^{+}(t_4)\dot{\underline{Z}}^{-}(t_5)]$$

$$= \int_0^\infty\int_0^\infty E[\underline{z}(0) \int_0^{t_3-\tau_f} \dot{\underline{z}}^{-}(\theta_3)d\theta_3\, \dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f$$

$$+ \int_0^\infty\int_0^\infty E[\underline{z}(0) \int_0^{t_2-\tau_r} \dot{\underline{z}}^{+}(\theta_2)d\theta_2 \int_0^{t_3-\tau_f} \dot{\underline{z}}^{-}(\theta_3)\, d\theta_3 \dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f$$

$$+ \int_0^\infty\int_0^\infty E[\underline{z}(0) \int_0^{t_1-\tau_r} \dot{\underline{z}}^{+}(\theta_1)d\theta_1 \int_0^{t_3-\tau_f} \dot{\underline{z}}^{-}(\theta_3)d\theta_3\, \dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r\, d\tau_f$$

$$+ \int_0^\infty\int_0^\infty E[\int_0^{t_1-\tau_r} \dot{\underline{z}}^{+}(\theta_1)d\theta_1 \int_0^{t_2-\tau_r} \dot{\underline{z}}^{+}(\theta_2)d\theta_2 \int_0^{t_3-\tau_f} \dot{\underline{z}}^{-}(\theta_3)d\theta_3\, \dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]$$

$$f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f. \tag{3.44}$$

Interchanging the order of expectations and integrations results in

$$E[\underline{Z}^{+}(t_1)\underline{Z}^{+}(t_2)\underline{Z}^{-}(t_3)\dot{\underline{Z}}^{+}(t_4)\dot{\underline{Z}}^{-}(t_5)] =$$

$$= \int_0^\infty\int_0^\infty\int_0^{t_3-\tau_f} E[\underline{z}(0)\dot{\underline{z}}^{-}(\theta_3)\dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\theta_3 d\tau_r d\tau_f$$

$$+ \int_0^\infty\int_0^\infty\int_0^{t_2-\tau_r}\int_0^{t_3-\tau_f} E[\underline{z}(0)\dot{\underline{z}}^{+}(\theta_2)\dot{\underline{z}}^{-}(\theta_3)\dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\theta_3 d\theta_2 d\tau_r d\tau_f$$

$$+ \int_0^\infty\int_0^\infty\int_0^{t_1-\tau_r}\int_0^{t_3-\tau_f} E[\underline{z}(0)\dot{\underline{z}}^{+}(\theta_1)\dot{\underline{z}}^{-}(\theta_3)\dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\theta_3 d\theta_1 d\tau_r d\tau_f$$

$$+\int_0^\infty\int_0^\infty\int_0^{t_1-\tau_r}\int_0^{t_2-\tau_r}\int_0^{t_3-\tau_f} E[\dot{\underline{z}}^{+}(\theta_1)\dot{\underline{z}}^{+}(\theta_2)\dot{\underline{z}}^{-}(\theta_3)\dot{\underline{z}}^{+}(t_4-\tau_r)\dot{\underline{z}}^{-}(t_5-\tau_f)]$$

$$f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\theta_3 d\theta_2 d\theta_1 d\tau_r d\tau_f. \tag{3.45}$$

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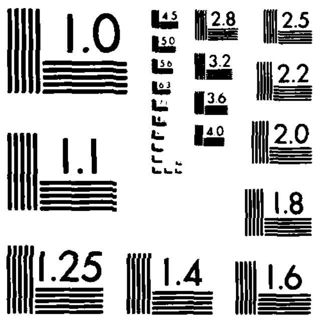

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Note in Eq.(3.45) that only binary variables (i.e. $\underline{z}(0)$ and derivatives of counting processes (i.e. $\dot{\underline{z}}^{+}(t)$ and $\dot{z}^{-}(t)$) occur in the higher order joint moments of the integrands. ∇

Evaluation of higher order joint moments in terms of the input switching variables, $\underline{X1}(t),\ldots,\underline{Xn}(t)$, is considered next.

A seven step procedure to be used in determining analytical expressions for $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$ in terms of the input switching variables and the time derivatives of their counting signals is discussed in Sec.2.3. This procedure is applicable to finding expressions for $\dot{\underline{z}}^{+}(\cdot)$ and $\dot{\underline{z}}^{-}(\cdot)$ such as occur in Eq.(3.45). However, now $\dot{\underline{z}}^{+}(\cdot)$ and $\dot{\underline{z}}^{-}(\cdot)$ are evaluated with various arguments (e.g., θ_1, θ_2, $t_4-\tau_r$, $t_5-\tau_f$). In order to perform arithmetic operations like addition and multiplication on terms involving $\underline{z}(0)$, $\underline{z}(0)$ must be expressed in the form of an arithmetic expression. This can be done by first obtaining the switching expression for $z(0)$ in terms of the input switching variables, $\underline{X1}(t),\ldots,\underline{Xn}(t)$. Next the switching expression is modified into the corresponding arithmetic expression utilizing the arithmetic expression theorem of Ch.2. The output higher order joint moments can then be obtained, as illustrated in the following example.

Example 3.9 Consider the physical NAND gate whose model is shown in Fig. 3.11. The inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ are assumed to be statistically independent. It is desired to evaluate the output second order joint moment $R_{\underline{Z}\dot{\underline{Z}}^{+}}(t_1,t_2) = E[\underline{Z}(t_1)\dot{\underline{Z}}^{+}(t_2)]$.

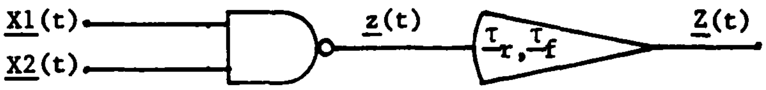


Fig. 3.11. Circuit model for a physical NAND gate.

The first step is to express the desired function, $R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2)$, in terms of the delay element input signal $\underline{z}(t)$ and the time derivatives $\dot{\underline{z}}^+(t)$ and $\dot{\underline{z}}^-(t)$. Decomposition of $\underline{Z}(t)$ into $\underline{Z}^+(t)-\underline{Z}^-(t)$ yields

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = E[\{\underline{Z}^+(t_1)-\underline{Z}^-(t_1)\}\dot{\underline{Z}}^+(t_2)]$$

$$= E[\underline{Z}^+(t_1)\dot{\underline{Z}}^+(t_2)] - E[\underline{Z}^-(t_1)\dot{\underline{Z}}^+(t_2)]. \tag{3.46}$$

Transforming in Eq.(3.46) the counting processes and their derivatives in terms of those at the delay element input results in

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = E[\underline{z}^+(t_1-\underline{\tau}_r)\dot{\underline{z}}^+(t_2-\underline{\tau}_r)] - E[\underline{z}^-(t_1-\underline{\tau}_f)\dot{\underline{z}}^+(t_2-\underline{\tau}_r)]. \tag{3.47}$$

Using Eq.(3.39) - (3.41), $\underline{z}^+(t_1-\underline{\tau}_r)$ and $\underline{z}^-(t_1-\underline{\tau}_f)$ are expressed in integral form. Substitution into Eq. (3.47) results in

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = E[\{\underline{z}(0)+\int_0^{t_1-\underline{\tau}_r}\dot{\underline{z}}^+(\theta)d\theta\}\dot{\underline{z}}^+(t_2-\underline{\tau}_r)]-E[\int_0^{t_1-\underline{\tau}_f}\dot{\underline{z}}^-(\theta)d\theta\,\dot{\underline{z}}^+(t_2-\underline{\tau}_r)].$$

$$= E[\underline{z}(0)\dot{\underline{z}}^+(t_2-\underline{\tau}_r)] + E[\int_0^{t_1-\underline{\tau}_r}\dot{z}^+(\theta)d\theta\,\dot{z}^+(t_2-\underline{\tau}_r)]-E[\int_0^{t_1-\underline{\tau}_f}\dot{\underline{z}}^-(\theta)d\theta\,\dot{\underline{z}}^+(t_2-\underline{\tau}_r)]. \tag{3.48}$$

Utilizing the conditional expectation theorem [18] yields

$$R_{\underline{Z}\,\dot{\underline{Z}}^{+}}(t_1,t_2) = \int_0^{\infty} E[\underline{z}(0)\dot{\underline{z}}^{+}(t_2-\tau_r)]f_{\underline{\tau}r}(\tau_r)d\tau_r$$

$$+\int_0^{\infty} E[\int_0^{t_1-\tau_r} \dot{\underline{z}}^{+}(\theta)d\theta\, \dot{\underline{z}}^{+}(t_2-\tau_r)]f_{\underline{\tau}r}(\tau_r)d\tau_r$$

$$-\int_0^{\infty}\int_0^{\infty} E[\int_0^{t_1-\tau_f} \dot{\underline{z}}^{-}(\theta)d\theta\dot{\underline{z}}^{+}(t_2-\tau_r)]f_{\underline{\tau}r\,\underline{\tau}f}(\tau_r,\tau_f)d\tau_r d\tau_f. \quad (3.49)$$

Because integration is a linear operation, the order of integration and expectation can be interchanged. Therefore,

$$R_{\underline{Z}\dot{\underline{Z}}^{+}}(t_1,t_2) = \int_0^{\infty} E[\underline{z}(0)\dot{\underline{z}}^{+}(t_2-\tau_r)]f_{\underline{\tau}r}(\tau_r)d\tau_r + \int_0^{\infty}\int_0^{t_1-\tau_r} E[\dot{\underline{z}}^{+}(\theta)\dot{z}^{+}(t_2-\tau_r)]f_{\underline{\tau}r}(\tau_r)d\theta d\tau_r$$

$$-\int_0^{\infty}\int_0^{\infty}\int_0^{t_1-\tau_f} E[\dot{\underline{z}}^{-}(\theta)\dot{\underline{z}}^{+}(t_2-\tau_r)]f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\theta d\tau_r d\tau_f. \quad (3.50)$$

Note that the integrands in Eq. (3.50) consist only of the binary signal $\underline{z}(0) = \underline{z}^{+}(0)$ and the derivatives of the counting processes $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$.

The next step is to express $R_{\underline{Z}\,\dot{\underline{Z}}^{+}}(t_1,t_2)$ in terms of the input switching signals, $\underline{X1}(t)$ and $\underline{X2}(t)$, and the derivatives of their counting signals.

The NAND switching operation is given by

$$\underline{z} = (\underline{X1}\cdot\underline{X2})' = \underline{X1}' \vee \underline{X2}'.$$

A sum of nonoverlapping implicants form is

$$\underline{z} = \underline{X1}' \vee \underline{X1} \cdot \underline{X2}'.$$

Applying the arithmetic expression theorem and using Eq.(3.23) (i.e., $\underline{Xk}' = 1 - \underline{Xk}$) yields

$$\underline{z} = (1 - \underline{X1}) + \underline{X1}(1 - \underline{X2}) = 1 - \underline{X1} + \underline{X1} - (\underline{X1})(\underline{X2}) = 1 - (\underline{X1})(\underline{X2}). \tag{3.51}$$

Differentiating Eq. (3.51) with respect to time, one obtains

$$\dot{\underline{z}}(t) = - \dot{\underline{X1}}(t)\ X2(t) - \dot{\underline{X2}}(t)\underline{X1}(t)$$

By comparison with Eq.(2.63), it follows that

$$\underline{B1}(t) = - \underline{X2}(t),\ \underline{B2}(t) = - \underline{X1}(t).$$

By definition of $\underline{Bk}^{+}(t)$ and $\underline{Bk}^{-}(t)$ (see Eq.(2.74)),

$$\underline{B1}^{+}(t) = 0,\ \underline{B1}^{-}(t) = \underline{X2}(t),\ \underline{B2}^{+}(t) = 0,\ B2^{-}(t) = \underline{X1}(t). \tag{3.52}$$

Substituting the results of Eq.(3.52) into Eqs.(2.79) results in

$$\dot{\underline{z}}^{+}(t) = \dot{\underline{X1}}^{-}(t)\underline{X2}(t) + \dot{\underline{X2}}^{-}(t)\underline{X1}(t),$$

$$\dot{\underline{z}}^{-}(t) = \dot{\underline{X1}}^{+}(t)\underline{X2}(t) + \dot{\underline{X2}}^{+}(t)\ \underline{X1}(t). \tag{3.53}$$

Substituting Eqs.(3.53) and $\underline{z}(0) = 1-\underline{X1}(0)\underline{X2}(0)$ (see Eq.(3.51)) into Eq.(3.50), $R_{\underline{Z}\,\dot{\underline{Z}}+}(t_1,t_2)$ becomes

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = \int_0^\infty E[\{1-\underline{X1}(0)\underline{X2}(0)\}\{\dot{\underline{X1}}^-(t_2-\tau_r)\underline{X2}(t_2-\tau_r)+\dot{\underline{X2}}^-(t_2-\tau_r)\underline{X1}(t_2-\tau_r)\}]f_{\underline{\tau}r}(\tau_r)d\tau_r$$

$$+\int_0^\infty\int_0^{t_1-\tau_r} E[\{\dot{\underline{X1}}^-(\theta)\underline{X2}(\theta)+\dot{\underline{X2}}^-(\theta)\underline{X1}(\theta)\}\{\dot{\underline{X1}}^-(t_2-\tau_r)\underline{X2}(t_2-\tau_r)$$

$$+\dot{\underline{X2}}^-(t_2-\tau_r)\underline{X1}(t_2-\tau_r)\}]f_{\underline{\tau}r}(\tau_r)d\theta d\tau_r$$

$$-\int_0^\infty\int_0^\infty\int_0^{t_1-\tau_f} E[\{\dot{\underline{X1}}^+(\theta)\underline{X2}(\theta)+\dot{\underline{X2}}^+(\theta)\underline{X1}(\theta)\}\{\dot{\underline{X1}}^-(t_2-\tau_r)\underline{X2}(t_2-\tau_r)$$

$$+\dot{\underline{X2}}^-(t_2-\tau_r)\underline{X1}(t_2-\tau_r)\}]f_{\underline{\tau}r\underline{\tau}f}(\tau_r,\tau_f)d\theta d\tau_r d\tau_f\ . \qquad (3.54)$$

Using the distributive law and the statistical independence of the inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ results in

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = \int_0^\infty \{E[\dot{\underline{X1}}^-(t_2-\tau_r)]E[\underline{X2}(t_2-\tau_r)] + E[\dot{\underline{X2}}^-(t_2-\tau_r)]E[\underline{X1}(t_2-\tau_r)]$$

$$-R_{\underline{X1}\dot{\underline{X1}}^-}(0,t_2-\tau_r)R_{\underline{X2X2}}(0,t_2-\tau_r)-R_{\underline{X1X1}}(0,t_2-\tau_r)R_{\underline{X2}\dot{\underline{X2}}^-}(0,t_2-\tau_r)\}f_{\underline{\tau}r}(\tau_r)d\tau_r$$

$$+\int_0^\infty\int_0^{t_1-\tau_r}\{R_{\dot{\underline{X1}}^-\dot{\underline{X1}}^-}(\theta,t_2-\tau_r)R_{\underline{X2X2}}(\theta,t_2-\tau_r)+R_{\underline{X1}\dot{\underline{X1}}^-}(t_2-\tau_r,\theta)R_{\underline{X2}\dot{\underline{X2}}^-}(\theta,t_2-\tau_r)$$

$$+R_{\underline{X1}\dot{\underline{X1}}^-}(\theta,t_2-\tau_r)R_{\underline{X2}\dot{\underline{X2}}^-}(t_2-\tau_r,\theta)+R_{\underline{X1X1}}(\theta,t_2-\tau_r)R_{\dot{\underline{X2}}^-\dot{\underline{X2}}^-}(\theta,t_2-\tau_r)\}f_{\underline{\tau}r}(\tau_r)d\theta d\tau_r$$

$$-\int_0^\infty\int_0^\infty\int_0^{t_1-\tau_f}\{R_{\dot{\underline{X1}}^+\dot{\underline{X1}}^-}(\theta,t_2-\tau_r)R_{\underline{X2X2}}(\theta,t_2-\tau_r)+R_{\underline{X1}\dot{\underline{X1}}^+}(t_2-\tau_r,\theta)R_{\underline{X2}\dot{\underline{X2}}^-}(\theta,t_2-\tau_r)$$

$$+ R_{\underline{X1}\,\dot{\underline{X1}}^-}(\theta,t_2-\tau_r)R_{\underline{X2}\,\dot{\underline{X2}}^+}(t_2-\tau_r,\theta)$$

$$+ R_{\underline{X1}\,\underline{X1}}(\theta,t_2-\tau_r)R_{\dot{\underline{X2}}^+\dot{\underline{X2}}^-}(\theta,t_2-\tau_r)\}f_{\underline{\tau r}\,\underline{\tau f}}(\tau_r,\tau_f)d\theta d\tau_r d\tau_f. \qquad (3.55)$$

Eq.(2.81) is applied to Eq.(3.55) to interchange order of differentiation and expectation, i.e. $E[\dot{\underline{Xk}}^-(\theta)] = \dot{E}[\underline{Xk}^-(\theta)]$. The same argument can be used in the correlation functions involving derivatives. For example,

$$R_{\underline{Xk}\,\dot{\underline{Xk}}^+}(t_1,t_2) = E[\underline{Xk}(t_1)\dot{\underline{Xk}}^+(t_2)] = \frac{\partial}{\partial t_2} E[\underline{Xk}(t_1)\underline{Xk}^+(t_2)]$$

$$= \frac{\partial}{\partial t_2}[R_{\underline{XkXk}^+}(t_1,t_2)]. \qquad (3.56)$$

The interchange can also be made when the correlation function involves two derivatives. For example,

$$R_{\dot{\underline{Xk}}^+\dot{\underline{Xk}}^-}(t_1,t_2) = E[\dot{\underline{Xk}}^+(t_1)\dot{\underline{Xk}}^-(t_2)] = \frac{\partial}{\partial t_2} E[\dot{\underline{Xk}}^+(t_1)\underline{Xk}^-(t_2)]$$

$$= \frac{\partial}{\partial t_2}[\frac{\partial}{\partial t_1} E[\underline{Xk}^+(t_1)\underline{Xk}^-(t_2)]] = \frac{\partial^2}{\partial t_1 \partial t_2} E[\underline{Xk}^+(t_1)\underline{Xk}^-(t_2)]$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2}[R_{\underline{Xk}^+\underline{Xk}^-}(t_1,t_2)]. \qquad (3.57)$$

Substituting Eqs.(2.81), (3.56), and (3.57) into Eq.(3.55), one finally obtains

$$R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2) = \int_0^\infty \{\dot{E}[\underline{X1}^-(t_2-\tau_r)]E[\underline{X2}(t_2-\tau_r)] + \dot{E}[\underline{X2}^-(t_2-\tau_r)]E[\underline{X1}(t_2-\tau_r)]$$

$$- \frac{\partial}{\partial t_2} [R_{\underline{X1X1}^-}(0,t_2-\tau_r)]R_{\underline{X2X2}}(0,t_2-\tau_r) - R_{\underline{X1X1}}(0,t_2-\tau_r) \frac{\partial}{\partial t_2} [R_{\underline{X2X2}^-}(0,t_2-\tau_r)]\}$$

$$f_{\underline{T}r}(\tau_r)d\tau_r$$

$$+ \int_0^\infty \int_0^{t_1-\tau_r} \{ \frac{\partial^2}{\partial\theta\partial t_2} [R_{\underline{X1}^-\underline{X1}^-}(\theta,t_2-\tau_r)]R_{\underline{X2X2}}(\theta,t_2-\tau_r) + \frac{\partial}{\partial\theta} [R_{\underline{X1X1}^-}(t_2-\tau_r,\theta)]$$

$$\frac{\partial}{\partial t_2} [R_{\underline{X2X2}^-}(\theta,t_2-\tau_r)]$$

$$+ \frac{\partial}{\partial t_2} [R_{\underline{X1X1}^-}(\theta,t_2-\tau_r)]\frac{\partial}{\partial\theta} [R_{\underline{X2X2}^-}(t_2-\tau_r,\theta)]$$

$$+ R_{\underline{X1X1}}(\theta,t_2-\tau_r) \frac{\partial^2}{\partial\theta\partial t_2} [R_{\underline{X2}^-\underline{X2}^-}(\theta,t_2-\tau_r)]\}f_{\underline{T}r}(\tau_r)d\theta d\tau_r$$

$$-\int_0^\infty \int_0^\infty \int_0^{t_1-\tau_f} \{ \frac{\partial^2}{\partial\theta\partial t_2} [R_{\underline{X1}^+\underline{X1}^-}(\theta,t_2-\tau_r)]R_{\underline{X2X2}}(\theta,t_2-\tau_r)$$

$$+ \frac{\partial}{\partial\theta}[R_{\underline{X1X1}^+}(t_2-\tau_r,\theta)] \frac{\partial}{\partial t_2} [R_{\underline{X2}\ \underline{X2}^-}(\theta,t_2-\tau_r)] + \frac{\partial}{\partial t_2} [R_{\underline{X1X1}^-}(\theta,t_2-\tau_r)]$$

$$\frac{\partial}{\partial\theta}[R_{\underline{X2X2}^+}(t_2-\tau_r,\theta)] + R_{\underline{X1X1}}(\theta,t_2-\tau_r) \frac{\partial^2}{\partial\theta\partial t_2} [R_{\underline{X2}^+\underline{X2}^-}(\theta,t_2-\tau_r)]\}f_{\underline{T}r\underline{T}f}(\tau_r,\tau_f)d\theta d\tau_r d\tau_f .$$

(3.58)

In the evaluation of $R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2)$ it is required to know all the expected values and correlation functions in Eq.(3.58). However, all correlation functions of the signal $\underline{Xk}(t)$ and its counting signals $\underline{Xk}^+(t)$ and $\underline{Xk}^-(t)$ can be expressed in terms of three basic correlation functions. A procedure for determining any correlation function in terms of the basic correlation functions is discussed next. ∇

There are 10 distinct correlation functions which may be defined for the signal $\underline{Y}(t)$, its complement $Y'(t)$ and its counting signals $\underline{Y}^+(t)$ and $\underline{Y}^-(t)$. These correlation function are listed below.

$$R_{\underline{YY}}(t_1,t_2),\ R_{\underline{YY}'}(t_1,t_2),\ R_{\underline{YY}^+}(t_1,t_2),\ R_{\underline{YY}^-}(t_1,t_2),\ R_{\underline{Y}'\underline{Y}'}(t_1,t_2),$$

$$R_{\underline{Y}'\underline{Y}^+}(t_1,t_2),R_{\underline{Y}'\underline{Y}^-}(t_1,t_2),R_{\underline{Y}^+\underline{Y}^+}(t_1,t_2),R_{\underline{Y}^+\underline{Y}^-}(t_1,t_2),\ R_{\underline{Y}^-\underline{Y}^-}(t_1,t_2).$$

However, using the decomposition $\underline{Y}(t) = \underline{Y}^+(t) - \underline{Y}^-(t)$, the first seven correlation functions on the above list may be expressed in terms of $R_{\underline{Y}^+\underline{Y}^+}(t_1,t_2)$,$R_{\underline{Y}^+\underline{Y}^-}(t_1,t_2)$, $R_{\underline{Y}^-\underline{Y}^-}(t_1,t_2)$, $E[\underline{Y}^+(t)]$, and $E[\underline{Y}^-(t)]$. The expected values $E[\underline{Y}^+(t)]$ and $E[\underline{Y}^-(t)]$ arise in correlation functions involving the complement $\underline{Y}'(t) = 1 - \underline{Y}(t)$. The following example illustrates the procedure.

<u>Example 3.10</u>. Consider the correlation function $R_{\underline{Y}'\underline{Y}^+}(t_1,t_2)$. Note that $\underline{Y}'(t) = 1-\underline{Y}(t) = 1 - \underline{Y}^+(t) + \underline{Y}^-(t)$. Therefore,

$$R_{\underline{Y}'\underline{Y}^+}(t_1,t_2) = E[\underline{Y}'(t_1)\underline{Y}^+(t_2)] = E[\{1-\underline{Y}^+(t_1)+\underline{Y}^-(t_1)\}\,\underline{Y}^+(t_2)] \tag{3.59}$$

Using the distributive law

$$R_{\underline{Y}'\underline{Y}^+}(t_1 t_2) = E[\underline{Y}^+(t_2)] - E[\underline{Y}^+(t_1)\underline{Y}^+(t_2)] + E[\underline{Y}^-(t_1)\underline{Y}^+(t_2)]. \tag{3.60}$$

By the definition of the correlation function (Eq.(3.5))

$$R_{\underline{Y}'\underline{Y}^+}(t_1,t_2) = E[\underline{Y}^+(t_2)] - R_{\underline{Y}^+\underline{Y}^+}(t_1,t_2) + R_{\underline{Y}^+\underline{Y}^-}(t_2,t_1). \tag{3.61}$$

Note that the right side of Eq.(3.61) contains the expected value $E[\underline{Y}^+(t_2)]$ and two of the three basic correlation functions. Expressions similar to Eq.(3.61) may be obtained for the other six correlation functions. ∇

Observe that Eq.(3.61) consists of three terms on the right hand side. In general, expressing correlation functions in terms of the basic three, $R_{\underline{Y}^+\underline{Y}^+}(t_1,t_2)$, $R_{\underline{Y}^+\underline{Y}^-}(t_1,t_2)$, $R_{\underline{Y}^-\underline{Y}^-}(t_1,t_2)$, and the expected values of the counting signals, $E[\underline{Y}^+(t)]$ and $E[\underline{Y}^-(t)]$, result in an increase in the number of terms involved in expressions similar to Eq.(3.58). For example, Eq.(3.58) consists of the sum of 12 terms. Expressing the various correlation functions in terms of the three basic ones and the expected values yields an expression consisting of the sum of 52 terms. Obviously, for efficiency of computation, summation in the intergrands should be performed before integration. For example, if this is done in Eq.(3.58) evaluation of $R_{\underline{Z}\dot{\underline{Z}}^+}(t_1,t_2)$ requires one single integration, one double integraion, and one triple integration.

It is next shown, by way of a simple example, how the basic

correlation functions, $R_{\underline{X}^+\underline{X}^+}(t_1,t_2)$, $R_{\underline{X}^+\underline{X}^-}(t_1,t_2)$, and $R_{\underline{X}^-\underline{X}^-}(t_1,t_2)$, can be obtained from the statistical properties of the signal $\underline{X}(t)$.

Example 3.11. Let $\underline{X}(t)$ be a 0,1 binary signal, containing a single pulse as shown in Fig. 3.12(a). The rise and fall times of the pulse, $\underline{u}_1$ and $\underline{d}_1$, respectively, are assumed to be independent random variables with the p.d.f's given in Fig. 3.12(b) and (c).

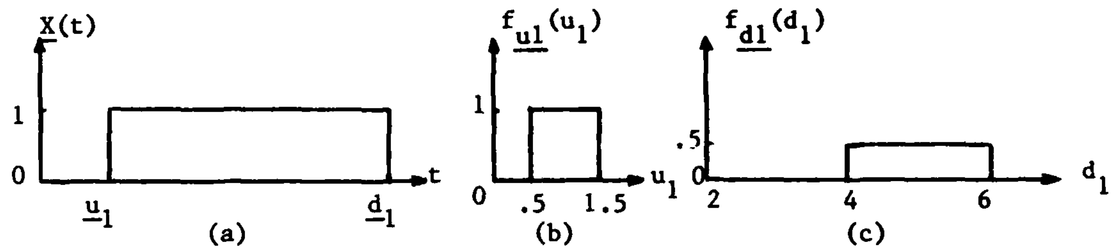


Fig. 3.12. a) random signal $\underline{X}(t)$, b)p.d.f of $\underline{u}_1$, c) p.d.f of $\underline{d}_1$.

It is desired to evaluate the basic correlation functions $R_{\underline{X}^+\underline{X}^+}(t_1,t_2)$, $R_{\underline{X}^+\underline{X}^-}(t_1,t_2)$, and $R_{\underline{X}^-\underline{X}^-}(t_1,t_2)$.

Fig. 3.13 shows the counting signals $\underline{X}^+(t)$ and $\underline{X}^-(t)$.

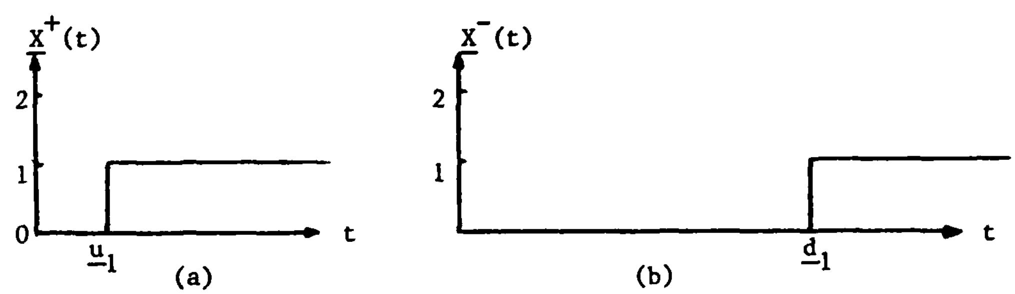


Fig. 3.13. The counting signals a) $\underline{X}^+(t)$ and b) $\underline{X}^-(t)$.

By definition

$$R_{\underline{X}^+\underline{X}^+}(t_1,t_2) = E[\underline{X}^+(t_1)\underline{X}^+(t_2)]. \tag{3.62}$$

Recall that $\underline{X}^+(t)$ is an integer valued signal. It follows that $\underline{X}^+(t_1)\cdot\underline{X}^+(t_2)$ is also integer valued. Therefore,

$$E[\underline{X}^+(t_1)\underline{X}^+(t_2)] = \sum_{k=-\infty}^{\infty} k \Pr\{\underline{X}^+(t_1)\underline{X}^+(t_2) = k\} . \tag{3.63}$$

In this case, $\underline{X}^+(t)$ takes on only the values 0 and 1. Hence the summation in Eq.(3.63) is reduced to

$$\begin{aligned} E[\underline{X}^+(t_1)\underline{X}^+(t_2)] &= 0\cdot\Pr\{\underline{X}^+(t_1)\underline{X}^+(t_2) = 0\} + 1\cdot\Pr\{\underline{X}^+(t_1)\underline{X}^+(t_2) = 1\} \\ &= \Pr\{\underline{X}^+(t_1)\underline{X}^+(t_2) = 1\}. \end{aligned} \tag{3.64}$$

Note that the product $\underline{X}^+(t_1)\underline{X}^+(t_2)$ takes on the value 1 only if both $\underline{X}^+(t_1) = 1$ and $\underline{X}^+(t_2) = 1$. Thus Eq.(3.64) may be written as

$$E[\underline{X}^+(t_1)\underline{X}^+(t_2)] = \Pr\{\underline{X}^+(t_1) = 1, \underline{X}^+(t_2) = 1\}. \tag{3.65}$$

Observing Fig. 3.13(a), one can see that $\underline{X}^+(t) = 1$ if and only if $\underline{u}_1 < t$. It follows that

$$\Pr\{\underline{X}^{+}(t_1) = 1, \underline{X}^{+}(t_2) = 1\} = \Pr\{\underline{u}_1 < t_1, \underline{u}_1 < t_2\}$$

$$= \begin{cases} \Pr\{\underline{u}_1 < t_1\}, & t_1 < t_2 \\ \Pr\{\underline{u}_1 < t_2\}, & t_2 < t_1 \end{cases} = \begin{cases} F_{\underline{u}1}(t_1) & t_1 < t_2 \\ F_{\underline{u}1}(t_2) & t_2 < t_1 \end{cases} . \qquad (3.66)$$

Combining Eqs.(3.62) and (3.66) results in

$$R_{\underline{X}^{+}\underline{X}^{+}}(t_1,t_2) = \begin{cases} F_{\underline{u}1}(t_1) & t_1 < t_2 \\ F_{\underline{u}1}(t_2) & t_2 < t_1 \end{cases} . \qquad (3.67)$$

The results in Eq.(3.67) are sketched in Fig. 3.14.

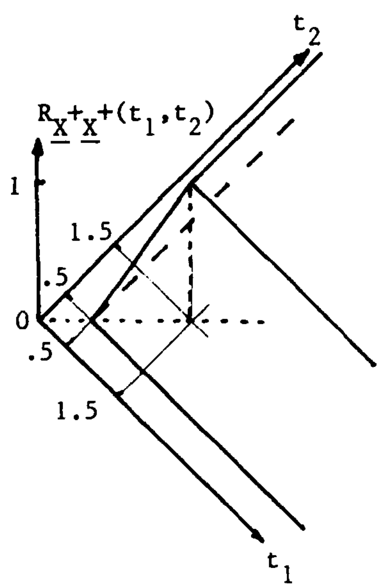


Fig. 3.14 The autocorrelation $R_{\underline{X}^{+}\underline{X}^{+}}(t_1,t_2)$

Similar derivations for the autocorrelation $R_{\underline{X}^-\underline{X}^-}(t_1,t_2)$ yield

$$R_{\underline{X}^-\underline{X}^-}(t_1,t_2) = \begin{cases} F_{\underline{d}1}(t_1) & t_1 < t_2 \\ \\ F_{\underline{d}1}(t_2) & t_2 < t_1 \end{cases} \tag{3.68}$$

This result is sketched in Fig. 3.15.

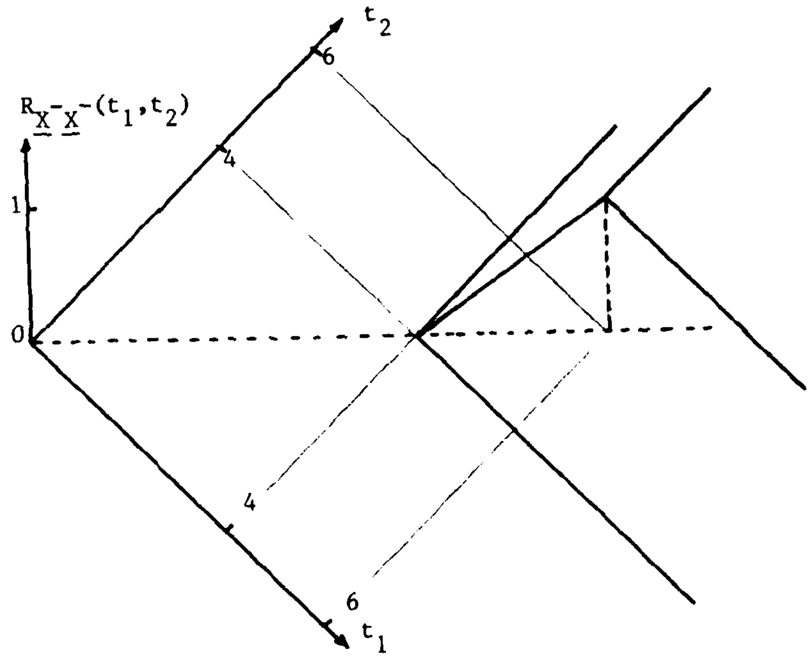


Fig. 3.15. The autocorrelation $R_{\underline{X}^-\underline{X}^-}(t_1,t_2)$.

The crosscorrelation function $R_{\underline{X}^+\underline{X}^-}(t_1,t_2)$ is obtained by similar reasoning.

$$R_{\underline{X}^+\underline{X}^-}(t_1,t_2) = E[\underline{X}^+(t_1)\underline{X}^-(t_2)] = 0\cdot \Pr\{\underline{X}^+(t_1)\ \underline{X}^-(t_2) = 0\} + 1\cdot \Pr\{\underline{X}^+(t_1)$$

$$= \Pr\{\underline{X}^+(t_1)\ \underline{X}^-(t_2) = 1\} = \Pr\{\underline{X}^+(t_1) = 1,\ \underline{X}^-(t_2) = 1\} \tag{3.69}$$

As in Eq.(3.66), Eq.(3.69) can be expressed in terms of the time instants $\underline{u}_1$ and $\underline{d}_1$.

$$R_{\underline{X}^+\underline{X}^-}(t_1,t_2) = \Pr\{\underline{u}_1 < t_1, \underline{d}_1 < t_2\} = F_{\underline{u}1\underline{d}1}(t_1,t_2) \tag{3.70}$$

Using the independence of $\underline{u}_1$ and $\underline{d}_1$, Eq. (3.70) becomes

$$R_{\underline{X}^+\underline{X}^-}(t_1,t_2) = F_{\underline{u}1}(t_1)F_{\underline{d}1}(t_2). \tag{3.71}$$

The result of Eq.(3.71) is sketched in Fig. 3.16.

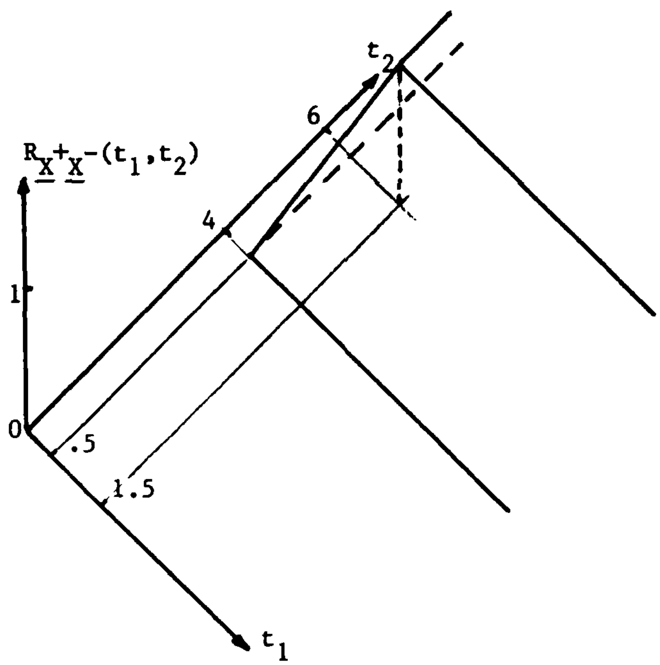


Fig. 3.16 The crosscorrelation $R_{\underline{X}^+\underline{X}^-}(t_1,t_2)$. ∇

The methods discussed so far for a single logic block can be utilized in analyzing combinational networks containing reconvergent fanout. As pointed out in Ch. 1, the output expected value E[Z(t)] can be used in evaluation of system performance. It was observed in Sec. 3.1 that higher order moments may arise when output expected values of networks containing reconvergent fanouts are considered. It was also shown and illustrated in this section how output higher order joint moments of a logic block can be obtained in terms of input higher order moments and the p.d.f's of the delays. An efficient method for evaluating the output expectation, E[Z(t)], of a general combinational network is described below.

In networks containing reconvergent fanouts it is possible that entire tree-like subnetworks may exist. Entire tree-like subnetworks are defined to be those in which there is only one path from any primary input Xk to an output Zj from the last network logic level. Entire tree-like subnetwork output expectations are evaluated by progressing from the primary inputs to the outputs, logic block by logic block, as described in Sec. 2.3. For the subnetworks including reconvergent fanouts, the output expectations are determined in terms of the higher order joint moments of the counting signals associated with the reconverging signals. However, only some joint moments of a given order may be necessary. In order to avoid unnecessary computations and storage of higher order joint moments, it is desirable to first

determine those moments which will be needed. This is done by 1) deriving the analytical expressions for $\dot{E}[\underline{z}^{+}(t)]$ and $\dot{E}[\underline{z}^{-}(t)]$ in terms of the reconverging signals and 2) expressing the resulting higher order moments of the reconverging signals in terms of those of the network inputs $\underline{X1}(t),\ldots,\underline{Xn}(t)$ and their counting signals. The procedure is illustrated in the following example.

Example 3.12. Consider the combinational network shown in Fig. 3.17. All gates are assumed to contain discriminating delay elements. Also, the inputs $\underline{X1}(t)$ and $\underline{X2}(t)$ are assumed to be statistically independent. Since there are two paths to the output for $\underline{X2}(t)$, the network contains a reconvergent fanout. It is desired to evaluate the expected value of the output $E[\underline{Z}(t)]$.

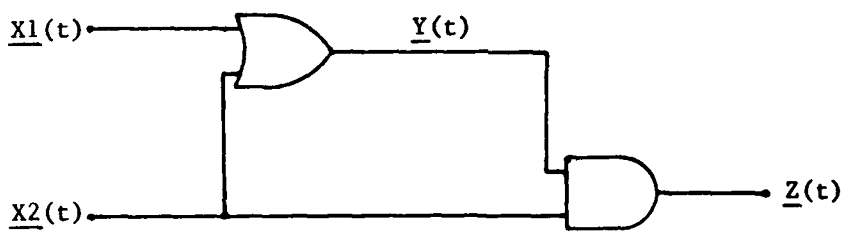


Fig. 3.17. Combinational network with a reconvergent fanout.

The model for the physical network in Fig. 3.17 is illustrated in Fig. 3.18. Observe the ideal logic output signals $\underline{y}(t)$ and $\underline{z}(t)$ and the discriminating delay elements with the random delays $\underline{\tau}_{r0}, \underline{\tau}_{f0}, \underline{\tau}_{rA}$ and $\underline{\tau}_{fA}$. Also, note that the network does not contain an entire tree-like subnetwork.

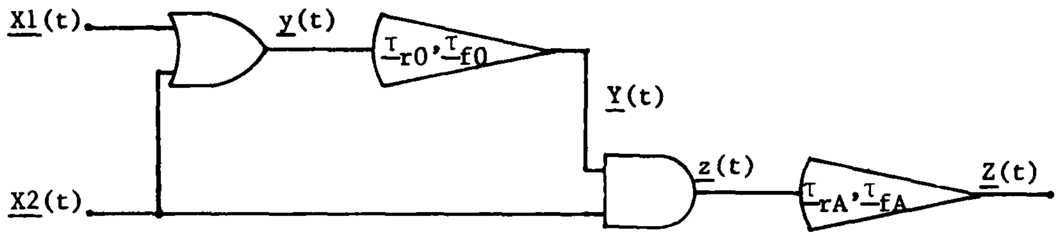


Fig. 3.18. Model for network in Fig. 3.17.

Because of the discriminating delays, the derivation for $E[\underline{Z}(t)]$ requires that $E[\underline{Z}^{+}(t)]$ and $E[\underline{Z}^{-}(t)]$ be determined separately. Then, by applying the expectation to Eq.(2.45), $E[\underline{Z}(t)]$ is given by

$$E[\underline{Z}(t)] = E[\underline{Z}^{+}(t)] - E[\underline{Z}^{-}(t)]. \tag{3.72}$$

Observe that the signal $\underline{X2}(t)$ splits and then reconverges at the node $\underline{z}$. To determine which higher order joint moments of $\underline{X2}(t)$ are needed, the arithmetic expressions for $\dot{\underline{z}}^{+}(t)$ and $\dot{\underline{z}}^{-}(t)$ in terms of the reconverging signal $\underline{X2}(t)$ are obtained first. Note that $\underline{z}(t)$ is the output of the ideal logic circuit performing the AND operation. Therefore,

$$\underline{z}(t) = \underline{X2}(t)\cdot\underline{Y}(t). \tag{3.73}$$

In Eq. (3.73) the right side can be considered to be in the form of a sum of nonoverlapping implicants. Differentiation of the corresponding arithmetic expression with respect to time yields

$$\dot{\underline{z}}(t) = \dot{\underline{X2}}(t)\underline{Y}(t) + \dot{\underline{Y}}(t)\underline{X2}(t). \qquad (3.74)$$

In order to place Eq.(3.74) in the form of Eq.(2.63), define

$$\dot{\underline{W1}}(t) = \dot{\underline{X2}}(t)$$

$$\dot{\underline{W2}}(t) = \dot{\underline{Y}}(t)$$

Then Eq.(3.74) can be written as

$$\dot{\underline{z}}(t) = \sum_{k=1}^{2} \dot{\underline{Wk}}(t)\underline{Bk}(t),$$

where

$$\underline{B1}(t) = \underline{Y}(t)$$

$$\underline{B2}(t) = \underline{X2}(t).$$

Note that $\underline{B1}(t)$ and $\underline{B2}(t)$ are in the arithmetic canonical form required by Eq.(2.72). With reference to Eq.(2.74), it follows that

$$\underline{B1}^{+}(t) = \underline{Y}(t),\ \underline{B1}^{-}(t) = 0,\ \underline{B2}^{+}(t) = \underline{X2}(t),\ \underline{B2}^{-}(t) = 0. \qquad (3.75)$$

Using Eq.(2.79), one obtains

$$\dot{\underline{z}}^{+}(t) = \dot{\underline{W1}}^{+}(t)\underline{Y}(t) + \dot{\underline{W2}}^{+}(t)\underline{X2}(t) = \dot{\underline{X2}}^{+}(t)\underline{Y}(t) + \dot{\underline{Y}}^{+}(t)\underline{X2}(t)$$

$$\dot{\underline{z}}^{-}(t) = \dot{\underline{W1}}^{-}(t)\underline{Y}(t) + \dot{\underline{W2}}^{-}(t)\underline{X2}(t) = \dot{\underline{X2}}^{-}(t)\underline{Y}(t) + \dot{\underline{Y}}^{-}(t)\underline{X2}(t). \qquad (3.76)$$

Recognizing that $\underline{Y}(t) = \underline{Y}^{+}(t) - \underline{Y}^{-}(t)$, Eq.(3.76) become

$$\dot{\underline{z}}^{+}(t) = \dot{\underline{X2}}^{+}(t)\{\underline{Y}^{+}(t) - \underline{Y}^{-}(t)\} + \dot{\underline{Y}}^{+}(t)\underline{X2}(t)$$

$$\dot{\underline{z}}^{-}(t) = \dot{\underline{X2}}^{-}(t)\{\underline{Y}^{+}(t) - \underline{Y}^{-}(t)\} + \dot{Y}^{-}(t)\underline{X2}(t). \tag{3.77}$$

Each of the equations in Eqs.(3.77) are now developed separately.

The derivative of the rise counting signal, $\dot{\underline{z}}^{+}(t)$, is considered first. Note that $\underline{Y}^{+}(t) = \underline{y}^{+}(t-\underline{\tau}_{r0})$ and $\underline{Y}^{-}(t) = \underline{y}^{-}(t-\underline{\tau}_{f0})$. Consequently

$$\dot{\underline{z}}^{+}(t) = \dot{\underline{X2}}^{+}(t)\ \{\underline{y}^{+}(t-\underline{\tau}_{r0}) - \underline{y}^{-}(t-\underline{\tau}_{f0})\} + \dot{\underline{y}}^{+}(t-\underline{\tau}_{r0})\underline{X2}(t). \tag{3.78}$$

The counting signals $\underline{y}^{+}(t-\underline{\tau}_{r0})$ and $\underline{y}^{-}(t-\underline{\tau}_{f0})$ are expressed in terms of their derivatives by means of Eqs.(3.39) - (3.41). Hence,

$$\begin{aligned}
\dot{\underline{z}}^{+}(t) &= \dot{\underline{X2}}^{+}(t)\ \{\underline{y}(0) + \int_0^{t-\underline{\tau}_{r0}} \dot{\underline{y}}^{+}(\theta)d\theta - \int_0^{t-\underline{\tau}_{f0}} \dot{\underline{y}}^{-}(\theta)d\theta\} + \dot{\underline{y}}^{+}(t-\underline{\tau}_{r0})\underline{X2}(t) \\
&= \dot{\underline{X2}}^{+}(t)\underline{y}(0) + \dot{\underline{X2}}^{+}(t)\int_0^{t-\underline{\tau}_{r0}} \dot{\underline{y}}^{+}(\theta)d\theta - \dot{\underline{X2}}^{+}(t)\int_0^{t-\underline{\tau}_{f0}} \dot{\underline{y}}^{-}(\theta)d\theta + \dot{\underline{y}}^{+}(t-\underline{\tau}_{r0})\underline{X2}(t) \\
&= \dot{\underline{X2}}^{+}(t)\underline{y}(0) + \int_0^{t-\underline{\tau}_{r0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{y}}^{+}(\theta)d\theta - \int_0^{t-\underline{\tau}_{f0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{y}}^{-}(\theta)d\theta + \dot{\underline{y}}^{+}(t-\underline{\tau}_{r0})\underline{X2}(t)
\end{aligned} \tag{3.79}$$

With respect to $\underline{y}(t)$ only $\underline{y}(0)$, $\underline{y}^{+}(t)$, and $\underline{y}^{-}(t)$ are involved in Eq.(3.79). These signals can now be written in terms of the primary input signals. In particular, $\underline{y}(t) = \underline{X1}(t) \vee \underline{X2}(t)$, and a corresponding arithmetic expression is $\underline{y}(t) = \underline{X1}(t) + \underline{X1}'(t)\underline{X2}(t)$. Therefore,

$$\underline{y}(0) = \underline{X1}(0) + \underline{X1}'(0)\underline{X2}(0) \tag{3.80}$$

and

$$\dot{\underline{y}}(t) = \dot{\underline{X1}}(t) + \dot{\underline{X1}}'(t)\underline{X2}(t) + \underline{X1}'(t)\dot{\underline{X2}}(t). \tag{3.81}$$

Using Eq.(2.62) in Eq. (3.81) yields

$$\begin{aligned} \dot{\underline{y}}(t) &= \dot{\underline{X1}}(t) - \dot{\underline{X1}}(t)\underline{X2}(t) + \dot{\underline{X2}}(t)\underline{X1}'(t) \\ &= \dot{\underline{X1}}(t)\,\{1 - \underline{X2}(t)\} + \dot{\underline{X2}}(t)X1'(t) \\ &= \dot{\underline{X1}}(t)\underline{X2}'(t) + \dot{\underline{X2}}(t)\underline{X1}'(t). \end{aligned} \tag{3.82}$$

Developing Eq.(3.82) into the form of Eq.(2.78), it follows that

$$\underline{B1}^{+}(t) = \underline{X2}'(t),\ \underline{B1}^{-}(t) = 0,\ \underline{B2}^{+}(t) = \underline{X1}'(t),\ \underline{B2}^{-}(t) = 0. \tag{3.83}$$

The use of Eqs.(3.83) in Eqs.(2.79) results in

$$\begin{aligned} \dot{\underline{y}}^{+}(t) &= \dot{\underline{X1}}^{+}(t)\underline{X2}'(t) + \dot{\underline{X2}}^{+}(t)\underline{X1}'(t), \\ \dot{\underline{y}}^{-}(t) &= \dot{\underline{X1}}^{-}(t)\underline{X2}'(t) + \dot{\underline{X2}}^{-}(t)\underline{X1}'(t). \end{aligned} \tag{3.84}$$

Substitution of Eqs.(3.80) and (3.84) into Eq. (3.79) yields

$$\dot{\underline{z}}^{+}(t) = \dot{\underline{X2}}^{+}(t)\underline{X1}(0) + \dot{\underline{X2}}^{+}(t)\underline{X1}'(0)\underline{X2}(0) + \int_0^{t-\underline{\tau}_{r0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{X1}}^{+}(\theta)\underline{X2}'(\theta)\, d\theta$$

$$+ \int_0^{t-\underline{\tau}_{r0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{X2}}^{+}(\theta)\underline{X1}'(\theta)d\theta$$

$$- \int_0^{t-\underline{\tau}_{f0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{X1}}^{-}(\theta)\underline{X2}'(\theta)d\theta - \int_0^{t-\underline{\tau}_{f0}} \dot{\underline{X2}}^{+}(t)\dot{\underline{X2}}^{-}(\theta)\underline{X1}'(\theta)d\theta$$

$$+ \dot{\underline{X1}}^{+}(t-\underline{\tau}_{r0})\underline{X2}'(t-\underline{\tau}_{r0})\underline{X2}(t)$$

$$+ \dot{\underline{X2}}^{+}(t-\underline{\tau}_{r0})\underline{X1}'(t-\underline{\tau}_{r0})\underline{X2}(t). \qquad (3.85)$$

The right side of Eq.(3.85) consists only of primary input signals. The expected value of $\dot{\underline{z}}^{+}(t)$ is now readily obtained. Making use of the statistical independence of the inputs $\underline{X1}(t)$ and $\underline{X2}(t)$, $E[\dot{\underline{z}}^{+}(t)]$ becomes

$$E[\dot{\underline{z}}^{+}(t)] = E[\dot{\underline{X2}}^{+}(t)]E[\underline{X1}(0)] + E[\underline{X1}'(0)]R_{\underline{X2}\dot{\underline{X2}}^{+}}(0,t) + \int_0^{t-\underline{\tau}_{r0}} E[\dot{\underline{X1}}^{+}(\theta)]R_{\underline{X2}'\dot{\underline{X2}}^{+}}(\theta,t)d\theta$$

$$+ \int_0^{t-\underline{\tau}_{r0}} E[\underline{X1}'(\theta)]R_{\dot{\underline{X2}}^{+}\dot{\underline{X2}}^{+}}(\theta,t)d\theta - \int_0^{t-\underline{\tau}_{f0}} E[\dot{\underline{X1}}^{-}(\theta)]R_{\dot{\underline{X2}}^{+}\underline{X2}'}(t,\theta)d\theta$$

$$- \int_0^{t-\underline{\tau}_{f0}} E[\underline{X1}'(\theta)]R_{\dot{\underline{X2}}^{+}\dot{\underline{X2}}^{-}}(t,\theta)d\theta$$

$$+ E[\dot{\underline{X1}}^{+}(t-\underline{\tau}_{r0})]R_{\underline{X2X2}'}(t,t-\underline{\tau}_{r0}) + E[\underline{X1}'(t-\underline{\tau}_{r0})]R_{\underline{X2}\,\dot{\underline{X2}}^{+}}(t,t-\underline{\tau}_{r0}). \qquad (3.86)$$

Next the order of differentiation and expectation is interchanged and the conditional expectation theorem is used to obtain

$$\dot{E}[\underline{z}^+(t)] = \dot{E}[\underline{X2}^+(t)]E[\underline{X1}(0)] + E[\underline{X1}'(0)]\frac{d}{dt}[R_{\underline{X2X2}}+(0,t)]$$

$$+ \int_0^\infty\int_0^{t-\tau_{r0}} \dot{E}[\underline{X1}^+(\theta)]\frac{\partial}{\partial t}[R_{\underline{X2}'\underline{X2}}+(\theta,t)]f_{\underline{\tau}r0}(\tau_{r0})d\theta d\tau_{r0}$$

$$+ \int_0^\infty\int_0^{t-\tau_{r0}} E[\underline{X1}'(\theta)]\frac{\partial^2}{\partial\theta dt}[R_{\underline{X2}^+\underline{X2}}+(\theta,t)]f_{\underline{\tau}r0}(\tau_{r0})d\theta d\tau_{r0}$$

$$- \int_0^\infty\int_0^{t-\tau_{f0}} \dot{E}[\underline{X1}^-(\theta)]\frac{\partial}{\partial t}[R_{\underline{X2}'\underline{X2}}+(\theta,t)]f_{\underline{\tau}f0}(\tau_{f0})d\theta d\tau_{f0}$$

$$- \int_0^\infty\int_0^{t-\tau_{f0}} E[\underline{X1}'(\theta)]\frac{\partial^2}{\partial t\partial\theta}[R_{\underline{X2}^+\underline{X2}}-(t,\theta)]f_{\underline{\tau}f0}(\tau_{f0})d\theta d\tau_{f0}$$

$$+ \int_0^\infty \dot{E}[\underline{X1}^+(t-\tau_{r0})]R_{\underline{X2X2}'}(t,t-\tau_{r0})f_{\underline{\tau}r0}(\tau_{r0})d\tau_{r0}$$

$$+ \int_0^\infty E[\underline{X1}'(t-\tau_{r0})]\frac{\partial}{\partial\theta}[R_{\underline{X2X2}}+(t,\theta-\tau_{r0})]\Big|_{\theta=t} f_{\underline{\tau}r0}(\tau_{r0})d\tau_{r0}. \quad (3.87)$$

Finally, the derivative of the expected value of the output rise counting signal, $\dot{E}[\underline{Z}^+(t)]$, is determined using Eq.(2.90). Specifically,

$$\dot{E}[\underline{Z}^+(t)] = \dot{E}[\underline{z}^+(t)]*f_{\underline{\tau}rA}(t). \quad (3.88)$$

A similar derivation for $\dot{E}[\underline{z}^-(t)]$ yields

$$\dot{E}[\underline{z}^-(t)] = \dot{E}[\underline{X2}^-(t)]E[\underline{X1}(0)]+E[\underline{X1}'(0)]\frac{d}{dt}[R_{\underline{X2X2}^-}(0,t)]$$

$$+ \int_0^\infty\int_0^{t-\tau_{r0}} \dot{E}[\underline{X1}^+(\theta)\frac{\partial}{\partial t}[R_{\underline{X2}'\underline{X2}^-}(\theta,t)]f_{\underline{\tau}\,r0}(\tau_{r0})d\theta d\tau_{r0}$$

$$+ \int_0^\infty\int_0^{t-\tau_{r0}} E[\underline{X1}'(\theta)]\frac{\partial^2}{\partial\theta\partial t}[R_{\underline{X2}^+\underline{X2}^-}(\theta,t)]f_{\underline{\tau}r0}(\tau_{r0})d\theta d\tau_{r0}$$

$$- \int_0^\infty\int_0^{t-\tau_{f0}} \dot{E}[\underline{X1}^-(\theta)]\frac{\partial}{\partial t}[R_{\underline{X2}'\underline{X2}^-}(\theta,t)]\, f_{\underline{\tau}f0}(\tau_{f0})d\theta d\tau_{f0}$$

$$- \int_0^\infty\int_0^{t-\tau_{f0}} E[\underline{X1}'(\theta)]\frac{\partial^2}{\partial\theta\partial t}[R_{\underline{X2}^-\underline{X2}^-}(\theta,t)]f_{\underline{\tau}f0}(\tau_{f0})d\theta d\tau_{f0}$$

$$+ \int_0^\infty \dot{E}[\underline{X1}^-(t-\tau_{f0})]R_{\underline{X2X2}'}(t,t-\tau_{f0})f_{\underline{\tau}f0}(\tau_{f0})d\tau_{f0}$$

$$+ \int_0^\infty E[\underline{X1}'(t-\tau_{f0})]\frac{\partial}{\partial\theta}[R_{\underline{X2}\ \underline{X2}^-}(t,\theta-\tau_{f0})]\Big|_{\theta=t} f_{\underline{\tau}f0}(\tau_{f0})d\tau_{f0}. \tag{3.89}$$

Once again, using Eq. (2.90)

$$\dot{E}[\underline{Z}^-(t)] = \dot{E}[\underline{z}^-(t)]*f_{\underline{\tau}fA}(t). \tag{3.90}$$

Finally, substituting Eqs.(3.88) and (3.90) into Eq.(2.88) and performing the integration with respect to time yield

$$E[\underline{Z}(t)] = E[\underline{Z}(0)] + \int_0^t \dot{E}[\underline{Z}^+(\theta)]d\theta - \int_0^t \dot{E}[\underline{Z}^-(\theta)]d\theta. \tag{3.91}$$

Because of Eqs.(3.87) and (3.89), note that E[$\underline{Z}$(t)] depends on the first and second order moments involving the reconverging signal $\underline{X2}$(t) and its counting signals. However, only the expected value of $\underline{X1}$(t) and its counting signals are required. ∇

In summary, the procedure for evaluating the output expected value of a combinational network with discriminating delay elements includes the following steps. First, the network is analyzed to identify reconvergent fanouts. The network is then separated into entire tree-like subnetworks and subnetworks consisting of the reconvergent fanouts. Entire tree-like subnetworks are defined to be those in which there is only one path from any primary input $\underline{Xk}$ to an output $\underline{Zj}$ from the last network logic level. Entire tree-like subnetwork output expectations are evaluated by progressing from the primary inputs to the outputs, logic block by logic block, as described in Sec. 2.3. For the subnetworks including reconvergent fanouts, the output expectations are first determined in terms of the higher order joint moments of the counting signals associated with the reconverging signals. This step was illustrated in Ex. 3.12. Next the higher order joint moments obtained for the reconverging signals are evaluated in terms of the higher order joint moments of the counting signals associated with the primary inputs. This step was illustrated in Ex. 3.9. Finally, the results of the last step are substituted into the expression for the output expected value, to yield an expression depending only on expectations and higher order joint moments of signals associated with primary inputs.

3.4 Approximations for Higher Order Moments of 0,1 Binary Processes

In their very significant contributions [15,16], Bass and Grundmann employed an interesting and easy to use approximation for the autocorrelation of a 0,1 binary process without discussing the conditions under which their approximation is valid. This approximation is developed in this section. In addition to discussing the limitations, it is also shown how the approximation can be generalized to higher order moments.

Consider a 0,1 binary stochastic signal $\underline{X}(t)$ whose autocorrelation function, $R_{\underline{XX}}(t_1,t_2)$, is required. In reference to Fig. 1.5, suppose every sample of $\underline{X}(t)$ has a minimum gap duration, denoted by υ_0, such that $\upsilon_0 \leq \underline{\upsilon}_i$; i=1,2,... Also suppose every sample of $\underline{X}(t)$ has a minimum pulse duration, denoted by δ_0, such that $\delta_0 \leq \underline{\delta}_j$; j = 1,2,...

Let $t_1 = t$ and $t_2 = t + \Delta$, where Δ is to be considered as a fixed constant during the discussion. A new 0,1 binary stochastic process is defined to be

$$\underline{Y}_\Delta(t) = \underline{X}(t)\underline{X}(t+\Delta). \tag{3.92}$$

The autocorrelation function of $\underline{X}(t)$ is simply the expectation of $\underline{Y}_\Delta(t)$. Specifically,

$$R_{\underline{XX}}(t,t+\Delta) = E[\underline{X}(t)\underline{X}(t+\Delta)] = E[\underline{Y}_\Delta(t)]. \tag{3.93}$$

Consider a sample function

$$Y_\Delta(t) = X(t)X(t+\Delta).$$

A rise transition in $Y_\Delta(t)$ exists only when a rise transition occurs in one of the factors of $Y_\Delta(t)$ while the other factor takes on the value 1. This is illustrated in Fig. 3.19. If $|\Delta| < \upsilon_0$, the i^{th} rise transition of $Y_\Delta(t)$ occurs at the i^{th} rise transition of $X(t)$ when $\Delta>0$ and at the i^{th} rise transition of $X(t+\Delta)$ when $\Delta<0$. In effect, the i^{th} rise transition in $Y_\Delta(t)$ occurs at the i^{th} rise transition of either $X(t)$ or $X(t+\Delta)$, depending on which comes last. It follows, for every sample function $Y_\Delta(t)$, that the rise counting signal is given by

$$Y_\Delta^+(t) = \begin{cases} X^+(t), & \Delta>0 \\ X^+(t+\Delta), & \Delta<0 \end{cases} = \min\{X^+(t),\ X^+(t+\Delta)\}. \tag{3.94}$$

Eq. (3.94) is illustrated in Fig. 3.19 for $0 < \Delta < \upsilon_0$.

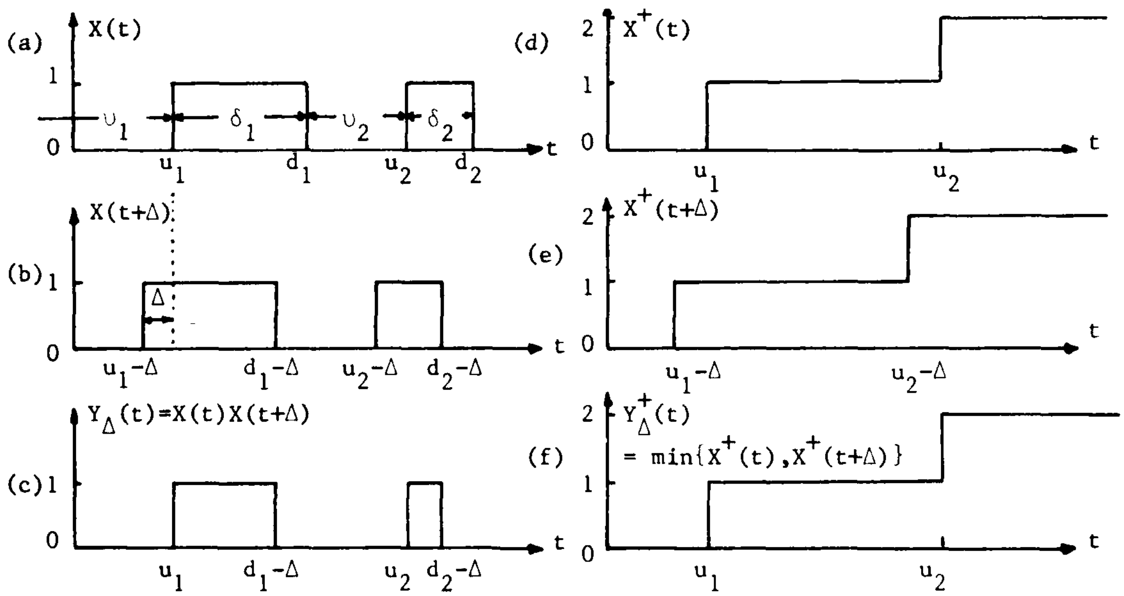


Fig. 3.19. Sample functions of a) $\underline{X}(t)$, b) $\underline{X}(t+\Delta)$, c) $\underline{Y}_\Delta(t)$ d) $\underline{X}^+(t)$, e) $\underline{X}^+(t+\Delta)$ and f) $\underline{Y}_\Delta^+(t)$ with the restriction that $0 < \Delta < \upsilon_0$.

It is important to emphasize that Eq.(3.94) does not hold when $|\Delta| > \upsilon_0$. This is illustrated in Fig. 3.20.

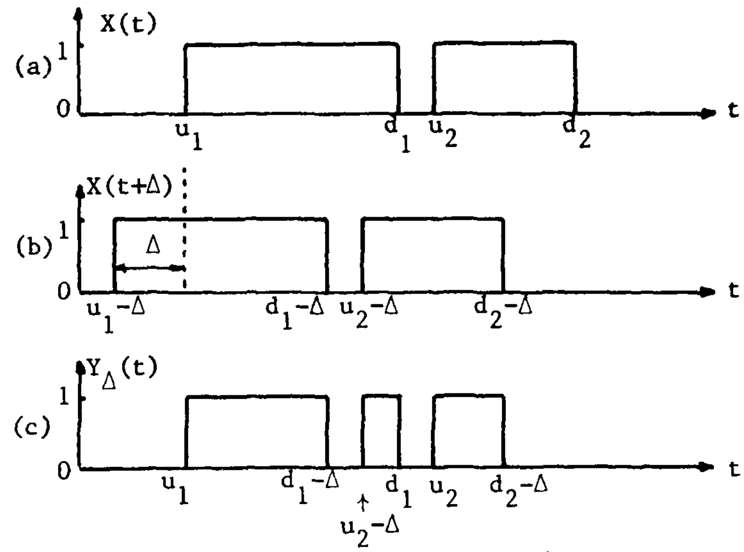


Fig. 3.20. Sample functions of a) $\underline{X}(t)$, b) $\underline{X}(t+\Delta)$, c) $\underline{Y}_\Delta(t)$ under the condition that $\upsilon_0 < \Delta$.

Even though $\Delta>0$, the second rise transition in $Y_\Delta(t)$ occurs at the second rise transition of $X(t+\Delta)$ as opposed to the second rise transition of $X(t)$. This is in contradiction to Eq.(3.94). The difficulty arises because $\Delta > \upsilon_0$.

Similar reasoning holds for the fall transitions of $Y_\Delta(t)$. The binary signal $Y_\Delta(t)$ has a fall transition if one of the factors, $X(t)$ or $X(t+\Delta)$, has a fall transition while the other takes on the value 1. This is illustrated in Fig. 3.21.

If $|\Delta| < \delta_0$ the j^{th} fall transition of $Y_\Delta(t)$ occurs at the j^{th} fall transition of $X(t+\Delta)$ when $\Delta > 0$ and at the j^{th} fall transition of $X(t)$ when $\Delta < 0$. In effect, the j^{th} fall transition of $Y_\Delta(t)$ occurs at the j^{th} fall transition of either $X(t)$ or $X(t+\Delta)$, depending on which comes first. It follows, for every sample function $Y_\Delta(t)$, that the fall

counting signal is given by

$$Y_{\Delta}^{-}(t) = \begin{cases} X^{-}(t+\Delta), & \Delta > 0 \\ X^{-}(t), & \Delta < 0 \end{cases} = \max\{X^{-}(t),\ X^{-}(t+\Delta)\} \tag{3.95}$$

Eq. (3.95) is illustrated in Fig. 3.21 for $0 < \Delta < \delta_0$.

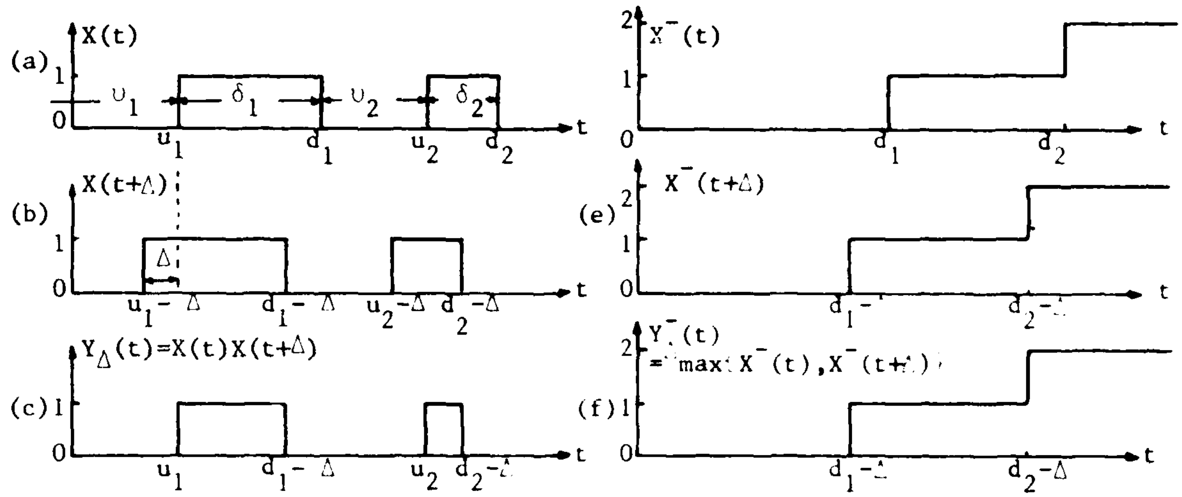


Fig. 3.21. Sample functions of a) $\underline{X}(t)$, b) $\underline{X}(t+\Delta)$, c) $\underline{Y}_{\Delta}(t)$, d) $\underline{X}^{-}(t)$, e) $\underline{X}^{-}(t+\Delta)$ and f) $\underline{Y}^{-}_{\Delta}(t)$ with the restriction that $0 < \Delta < \delta_0$.

Eq. (3.95) is not valid when $\Delta > \delta_0$. This is illustrated in Fig. 3.22.

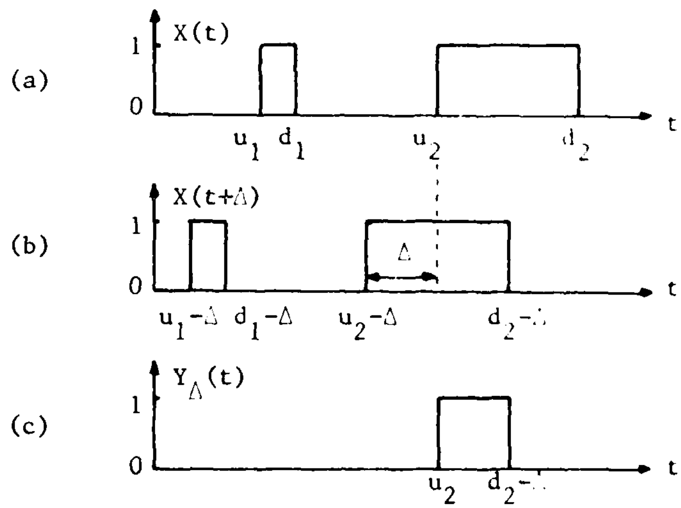


Fig. 3.22. Sample functions of a) $\underline{X}(t)$, b) $\underline{X}(t+\Delta)$, c) $\underline{Y}_{\Delta}(t)$ under the condition that $\delta_0 < \Delta$.

Even though $\Delta > 0$, a fall transition does not occur in $Y_\Delta(t)$ when the first fall transition occurs in $X(t+\Delta)$. Eq.(3.95) is violated because $\Delta > \delta_0$.

Combining the piecewise results of Eqs.(3.94) and (3.95) in accordance with Eq.(1.5), an expression for $Y_\Delta(t)$ is given by

$$Y_\Delta(t) = Y_\Delta^+(t) - Y_\Delta^-(t) = \begin{cases} X^+(t) - X^-(t+\Delta) & \Delta>0 \\ X^+(t+\Delta) - X^-(t)\ , & \Delta<0 \end{cases}\ ,\ |\Delta| < \min\{\upsilon_0, \delta_0\}. \quad (3.96)$$

An equivalent expression, using the min(•) and max (•) operations from Eqs(3.94) and (3.95), is

$$Y_\Delta(t) = \min\{X^+(t), X^+(t+\Delta)\} - \max\{X^-(t), X^-(t+\Delta)\}$$

$$|\Delta| < \min\{\upsilon_0, \delta_0\}. \quad (3.97)$$

Observe that the choice of the upper or lower equation in Eq. (3.96) is determined only by the sign of Δ. This choice holds for each time instant t and for all sample functions of the process $\underline{Y}_\Delta(t)$. It follows that Eqs.(3.96) and (3.97) hold for the stochastic process $\underline{Y}_\Delta(t)$. Specifically,

$$\underline{Y}_\Delta(t) = \left\{ \begin{matrix} \underline{X}^+(t) - \underline{X}^-(t+\Delta)\ , & \Delta>0 \\ \underline{X}^+(t+\Delta) - \underline{X}^-(t)\ , & \Delta<0 \end{matrix} \right\} = \min\{\underline{X}^+(t), \underline{X}^+(t+\Delta)\} - \max\{\underline{X}^-(t), \underline{X}^-(t+\Delta)\},$$

$$|\Delta| < \min\{\upsilon_0, \delta_0\} \quad (3.98)$$

where the $\min\{\cdot\}$ operation is applied to each of the sample function pairs from $\underline{X}^+(t)$ and $\underline{X}^+(t+\Delta)$ to generate the random process $\underline{Y}_\Delta^+(t)$ and $\underline{Y}^-_\Delta(t)$ is obtained in a similar manner using the $\max\{\cdot\}$ operation on sample function pairs from $\underline{X}^-(t)$ and $\underline{X}^-(t+\Delta)$. The expected value $E[\underline{Y}_\Delta(t)]$, is given by taking the expectation of the first part in Eq.(3.98) as

$$E[\underline{Y}_\Delta(t)] = \begin{cases} E[\underline{X}^+(t)] - E[\underline{X}^-(t+\Delta)], & \Delta>0 \\ E[\underline{X}^+(t+\Delta)] - E[\underline{X}^-(t)], & \Delta<0 \end{cases}, \quad |\Delta|<\min\{\upsilon_0,\delta_0\}. \quad (3.99)$$

Note that $E[\underline{X}^+(t)]$ and $E[\underline{X}^-(t)]$ are nondecreasing time functions since their respective time derivatives, $\dot{E}[\underline{X}^+(t)]$ and $\dot{E}[\underline{X}^-(t)]$, are nonnegative [15, p.18].

It follows that

$$\begin{aligned} E[\underline{X}^+(t)] &< E[\underline{X}^+(t+\Delta)] \;, \; \Delta > 0 \\ E[\underline{X}^+(t+\Delta)] &< E[\underline{X}^+(t)] \;, \; \Delta < 0 \end{aligned} \quad (3.100)$$

and

$$\begin{aligned} E[\underline{X}^-(t+\Delta)] &> E[\underline{X}^-(t)] \;, \; \Delta > 0 \\ E[\underline{X}^-(t)] &> E[\underline{X}^-(t+\Delta)] \;, \; \Delta < 0 \;. \end{aligned} \quad (3.101)$$

Consequently, $E[\underline{Y}_\Delta(t)]$ can be written as

$$E[\underline{Y}_\Delta(t)] = \min\{E[\underline{X}^+(t)],E[\underline{X}^+(t+\Delta)]\} - \max\{E[\underline{X}^-(t)],E[\underline{X}^-(t+\Delta)]\}, |\Delta|<\min\{\upsilon_0,\delta_0\}. \quad (3.102)$$

Finally, by referring to Eq.(3.92) and letting $t = t_1$ and $t + \Delta = t_2$,

Eq.(3.102) becomes

$$E[\underline{X}(t_1)\underline{X}(t_2)] = R_{\underline{XX}}(t_1,t_2) = \min\{E[\underline{X}^+(t_1)],E[\underline{X}^+(t_2)]\}$$

$$- \max\{E[\underline{X}^-(t_1)],E[\underline{X}^-(t_2)]\},$$

$$\text{provided } |t_1-t_2| < \min\{\upsilon_0,\delta_0\}. \tag{3.103}$$

The result of Eq. (3.103) is readily generalized to higher order moments. Consider the n-fold product

$$Y(t) = X(t)X(t+\Delta_1)\ \ldots X(t+\Delta_{n-1}) \tag{3.104}$$

where X(t) is a 0,1 binary signal and Δ_k, k = 1, ..., (n-1), may be either positive or negative constants. A typical scatter of the points t, $t+\Delta_1$,..., $t+\Delta_{n-1}$ is sketched in Fig. 3.23.

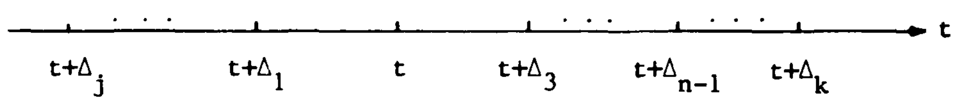


Fig. 3.23. Scatter of the points t, $t+\Delta_1$,..., $t+\Delta_{n-1}$, $0 \le j, k \le (n-1)$.

Denote

$$\Delta_M = \max\ \{\Delta_k,\ 0\}$$

$$\Delta_m = \min\ \{\Delta_k,\ 0\} \qquad ;\ k = 1,\ldots,(n-1). \tag{3.105}$$

The duration of the time interval over which the points t, $t+\Delta_1,\ldots,t+\Delta_{n-1}$ are scattered is specified by

$$T = \Delta_M - \Delta_m. \tag{3.106}$$

Note that $Y(t)$ equals 1 only if all n factors, $X(t), X(t+\Delta_1),\ldots,X(t+\Delta_{n-1})$, take on the value 1. Given a set of time shifts $\{\Delta_k\}$, such that

$$T < \min\{\upsilon_0,\delta_0\}, \tag{3.107}$$

the i^{th} rise transition of $Y(t)$ occurs at the i^{th} rise transition of either $X(t)$, $X(t+\Delta_1),\ldots X(t+\Delta_{n-1})$, depending upon which one occurs last. Similarly, the j^{th} fall transition of $Y(t)$ occurs at the j^{th} fall transition of either $X(t)$, $X(t+\Delta_1),\ldots,X(t+\Delta_{n-1})$, depending upon which one occurs first. It follows that

$$Y^+(t) = \min\{X^+(t),X^+(t+\Delta_1),\ldots,X^+(t+\Delta_{n-1})\}$$

$$Y^-(t) = \max\{X^-(t),X^-(t+\Delta_1),\ldots,X^-(t+\Delta_{n-1})\}, \tag{3.108}$$

provided $T < \min\{\upsilon_0,\delta_0\}$.

Consequently,

$$Y(t) = Y^+(t) - Y^-(t) = \min\{X^+(t),X^+(t+\Delta_1),\ldots,X^+(t+\Delta_{n-1})\}$$
$$- \max\{X^-(t),X^-(t+\Delta_1),\ldots,X^-(t+\Delta_{n-1})\} ;\ T < \min\{\upsilon_0,\delta_0\}. \tag{3.109}$$

Extending the same reasoning to each sample function of the stochastic process

$$\underline{Y}(t) = \underline{X}(t)\underline{X}(t+\Delta_1) \ \ldots \ \underline{X}(t+\Delta_{n-1}) \tag{3.110}$$

where $\underline{X}(t)$ is a stochastic 0,1 binary process and Δ_k's are constants, it follows that

$$\underline{Y}(t) = \min\{\underline{X}^+(t),\underline{X}^+(t+\Delta_1),\ldots,\underline{X}^+(t+\Delta_{n-1})\} - \max\{\underline{X}^-(t),\underline{X}^-(t+\Delta_1),\ldots,\underline{X}^-(t+\Delta_{n-1})\}$$

$$T < \min\{\upsilon_0,\delta_0\}. \tag{3.111}$$

The min $\{\cdot\}$ and max $\{\cdot\}$ operations yield $\underline{Y}^+(t)$ and $\underline{Y}^-(t)$, respectively, as indicated in the discussion following Eq.(3.98). Because the Δ_k's are constants and the counting signals are nondecreasing waveforms, it follows that

$$\underline{Y}^+(t) = \min\{\underline{X}^+(t),\underline{X}^+(t+\Delta_1),\ldots,\underline{X}^+(t+\Delta_{n-1})\} = \underline{X}^+(t+\Delta_m),$$

$$\underline{Y}^-(t) = \max\{\underline{X}^-(t),\underline{X}^-(t+\Delta_1),\ldots,\underline{X}^-(t+\Delta_{n-1})\} = \underline{X}^-(t+\Delta_M). \tag{3.112}$$

Eq. (3.111) becomes

$$\underline{Y}(t) = \underline{X}^+(t+\Delta_m) - \underline{X}^-(t+\Delta_M). \tag{3.113}$$

The expected value of $\underline{Y}(t)$ is then given by

$$E[\underline{Y}(t)] = E[\underline{X}^+(t+\Delta_m)] - E[\underline{X}^-(t+\Delta_M)]. \tag{3.114}$$

Recognizing that

$$E[\underline{X}^+(t+\Delta_m)] \leq E[\underline{X}^+(t+\Delta_k)],$$

$$E[\underline{X}^-(t+\Delta_M)] \geq E[\underline{X}^-(t+\Delta_k)] \quad k = 1,\ldots,(n-1) \tag{3.115}$$

Eq. (3.114) can be written as

$$E[\underline{Y}(t)] = \min\{E[\underline{X}^+(t)], E[\underline{X}^+(t+\Delta_1)], \ldots, E[\underline{X}^+(t+\Delta_{n-1})]\}$$

$$- \max\{E[\underline{X}^-(t), E[\underline{X}^-(t+\Delta_1)], \ldots, E[\underline{X}^-(t+\Delta_{n-1})]\}; T < \min\ (\upsilon_0, \delta_0). \quad (3.116)$$

Letting $t = t_1$, $t+\Delta_1 = t_2, \ldots, t + \Delta_{n-1} = t_n$, it follows that

$$T = \max\{t_k\} - \min\ \{t_k\}\ ;\ k=1,\ldots,n \quad (3.117)$$

and

$$E[\underline{X}(t_1)\ldots\underline{X}(t_n)] = \min\{E[\underline{X}^+(t_1)], \ldots, E[\underline{X}^+(t_n)]\} - \max\{E[\underline{X}^-(t_1)], \ldots, E[\underline{X}^-(t_n)]\},$$

$$T < \min\{\upsilon_0, \delta_0\}. \quad (3.118)$$

Eq. (3.118) is the generalization of Eq.(3.103) to n^{th} order moments.

The result in Eq.(3.103) is the approximation used by Grundmann in [15]. The derivation and limitations of this approximation have been presented in this section. Unfortunately, the restriction $T < \min\{\upsilon_0, \delta_0\}$ cannot always be met. In fact, in general, the existence of nonzero values of υ_0 and/or δ_0 is not guaranteed. In such cases use of the approximation results in some error. If the error is unacceptable, the higher order statistics introduced in this chapter must be used.

4. APPROXIMATE MODELS FOR LARGE LOGIC BLOCKS

4.1 Introduction

As was seen in the previous chapters, calculations required for the determination of output expected values may be very long and complicated, even for simple combinational circuits. One way for reducing the amount of computations and memory required for evaluation of complicated combinational circuits is to subdivide the circuit into large logic blocks. Each large logic block is modeled as a single entity and the entire circuit is analyzed by taking into account the interconnections of the large logic blocks. In general, a large logic block will contain many gates involving several levels of logic. In addition, the paths connecting the various inputs and outputs may differ significantly in terms of the number and types of gates encountered. For example, one input may be tied directly to the last gate of the logic block whereas other inputs may go through several levels of logic. As a result, insertion of delays into a simplified model of a large logic block is extremely difficult.

Several approaches have been suggested by Grundmann [15, Ch. 5]. The most accurate appears to be one in which a different delay is assigned at each input for every output affected by that input. Even though this approach does not take into account reconvergent fanouts, it is still extremely complicated because a large logic block with n inputs and m outputs requires that 2(n)(m) delays be specified in order

to account for the delays associated with the rise and fall transitions of all possible input-output pairs. Another approach is to simply place a single delay at each input irrespective of the number of outputs affected by that input. In this way a separate delay is attached to each input. However, a disadvantage is that the same delay is attributed to every path connected to a particular input even though the paths may have different delays. A third approach is to assign delays both at each input and at each output. This requires that 2(n+m) delays be assigned for a large logic block with n inputs and m outputs. The assignment is complicated by the fact that the delays are likely to be strongly dependent. The last approach suggested by Grundmann is to simply assign a delay to each output of the large logic block. A disadvantage of this approach is that the same delay is assigned to every path connected to a particular output even though the paths may have different delays. Although Grundmann proposed the various approaches described above, he did not discuss any strategies for characterizing the delays.

In this chapter, attention is devoted to the approach where a delay is assigned to each output of a large logic block. This model is referred to as the simple output delay model. The justification for this choice is discussed next. One factor motivating this choice is its relative simplicity. As seen in the examples of the previous chapters, the complexity of the analysis increases drastically as the number of delays increases. In addition, it has been observed in computer simulations

[3] and in experimental investigations [2, 23] that digital circuits are most sensitive to interfering signals injected into their outputs. In general, interference in integrated circuits is picked up in long wires and conductors. When two logic blocks are connected by a long-wire, the output gate in the logic block whose output is connected to the wire suffers the greatest delays due to interference on the wire (see Fig. 4.1). Consequently, the physical situation with respect to

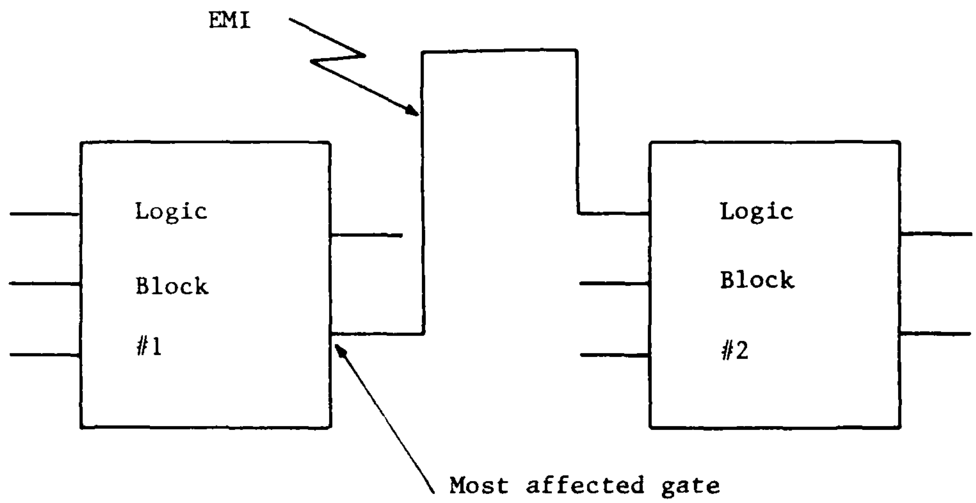


Fig. 4.1. Layout of a logic circuit subjected to EMI.

EMI is best modeled by associating the delay with the output. In addition, the technique developed in chapters 2 and 3 are readily applied to the simple output delay model.

Several considerations arise when subdividing a complicated combinational circuit into large logic blocks. Because the same delay is assigned to every path connected to a particular output, it is desirable to select large logic blocks such that the number of gates encountered in each path from any input to any output is approximately the same.

There is also a limit to how large a logic block should be. It is desirable to choose the logic block to be as large as possible in order to reduce the number of blocks and simplify the analysis. However, the spreads in the delays increase with the number of logic levels. Once the spreads exceed the minimum time duration between transitions in the input signals, some pulses or gaps that actually occur in the physical circuit are not predicted by the ideal logic portion of the model. In such cases, adding delay to the output of the ideal logic block results in errors. The phenomenon is illustrated in Example 1.1. Because the difference in the delays assigned to gates 01 and 02 exceeds the time duration between the transitions of the input signals, an output pulse is generated. If the ideal logic, without delay, is performed first, the output pulse is not generated and adding delay to the output of the ideal logic yields an error. To avoid such errors, it is recommended that the size of large logic blocks be chosen such that the difference between the maximum and minimum delays for all paths within the block be less than the minimum time duration between transitions in the input signals,where durations from one input to another are considered as well as durations within each input. This is a conservative recommendation in the sense that some error may be acceptable in order to simplify the analysis by increasing block size.

Obviously, error is encountered when large logic blocks are modeled by assigning single delays to each output. Can various strategies for characterizing the delays be employed to trade between simplicity and the degree of error? This is discussed next.

4.2 Output Delay Characterization

A difficult problem that arises in the simple output delay model deals with characterizing the output delays. Focusing attention on a particular output, there is usually more than one block input leading to that output. Also, more than one path may exist between an input and the output. (In such a case the network is said to have reconvergent fanout). Since different paths are likely to have different delays, any strategy for characterizing the output delay pair $\{\underline{\tau}_r, \underline{\tau}_f\}$ results in an approximation to the large logic block. The problem is further complicated because delay observed at an output, in general, depends on values of the inputs. This latter point is illustrated in Example 4.1.

Example 4.1 Consider the circuit shown in Fig. 4.2. This is a 2

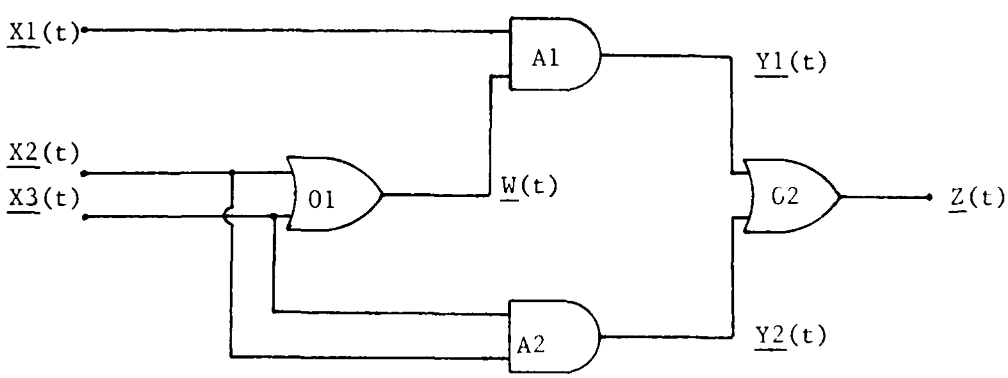


Fig. 4.2. A 2 out of 3 majority logic circuit.

out of 3 majority logic circuit. It will be investigated throughout this chapter, including the computer simulations. In this example, attention

is focused on a rise transition propagating from the input node X3 to the output node Z. Let $X1(t_1)=1$, $X2(t_1)=0$. In this case a rise transition in X3(t) which occurs at $t=t_1$ propagates through the gates 01, A1, and 02, and characterization of the output delay is determined by the p.d.f.'s associated with the rise propagation delays of these three gates. Next, let $X1(t_1)=0$ and $X2(t_1)=1$. Now a rise transition in X3(t) at $t=t_1$ propagates through the gates A2 and 02, and characterization of the output delay is determined by p.d.f's associated with the rise propagation delays of these two gates. It is seen that the delay characteristics observed at the output can depend on the input values.

The difficulty illustrated by Ex. 4.1 can be overcome by assigning p.d.f's to the output which are dependent on the inputs. However, this becomes very cumbersome. Consequently, effort in this chapter is devoted to developing strategies for assigning p.d.f.'s at the outputs which are not input dependent.

Four strategies for characterizing the output random delays by assigning input indendent p.d.f.'s are considered next.

1) Longest path delay In this strategy the longest path is used to characterize the output rise and fall random delays. By definition, the longest path is that which contains the largest number of gates. If two or more paths contain the largest number of gates, the longest path may be selected by applying a suitable criterion such as the sum of mean delays or the sum of maximum delays along each path.

The characterization of the output rise and fall transition delays must be handled separately. First, the logic block inputs are selected such that transition propagates from the path input to output. As the transition propagates, it is noted whether a rise or fall transition occurs at the output of each gate along the path. The total path delay for the rise or fall at the output equals the sum of the rise and fall propagation delays experienced along the path. Assuming statistical independence from gate to gate, the p.d.f. of the total delay is obtained by convolving the p.d.f's of the delays in the sum. This procedure is repeated so that p.d.f's are obtained for both rise and fall transition delays at the path output.

Example 4.2. Consider the circuit shown in Fig. 4.2. The longest path in the circuit includes the gates 01, A1 and 02. It is desired to obtain p.d.f.'s for both rise and fall transition delays at the longest path output.

Assume the logic block inputs have been selected such that a rise transition is generated at the output of the gate 01 and propagates to the output $\underline{Z}(t)$ along the longest path. By inspection, a rise transition at the input to gate A1 will create a rise transition at its output. This rise transition will, in turn, produce a rise transition at the outpu of gate 02. Denote the rise transition delays for gates 01, A1 and 02 and for the longest path by $\underline{\tau}_{r01}$, $\underline{\tau}_{rA1}$, $\underline{\tau}_{r02}$ and $\underline{\tau}_{rT}$, respectively. It follows that

$$\underline{\tau}_{rT} = \underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02} \ . \tag{4.1}$$

Similarly, let the fall transition delays be denoted by $\underline{\tau}_{f01}$, $\underline{\tau}_{fA1}$, $\underline{\tau}_{f02}$ and $\underline{\tau}_{fT}$. In an analogous manner, it is readily shown that

$$\underline{\tau}_{fT} = \underline{\tau}_{f01} + \underline{\tau}_{fA1} + \underline{\tau}_{f02}. \tag{4.2}$$

Let the delays of the same type of gates be identically distributed. Assuming statistical independence among delays of different gates, the p.d.f.'s of the total delays are given by

$$f_{\underline{\tau rT}}(\tau) = f_{\underline{\tau r0}}(\tau)*f_{\underline{\tau rA}}(\tau)*f_{\underline{\tau r0}}(\tau),$$

and

$$f_{\underline{\tau fT}}(\tau) = f_{\underline{\tau f0}}(\tau)*f_{\underline{\tau fA}}(\tau)*f_{\underline{\tau f0}}(\tau). \tag{4.3}$$

If the entire circuit of Fig. 4.2 is considered as a large logic block, the internal delays are ignored and a single discriminating delay element is assigned to the output node $\underline{Z}$. This delay element has $\underline{\tau}_{rT}$ and $\underline{\tau}_{fT}$ as its rise and fall transition delays, respectively, with p.d.f.'s given by Eqs.(4.3). ∇

2) Mean path delay. In this strategy, an expression is obtained for the arithmetic mean delay of all possible propagation paths from the inputs to the output. The arithmetic mean delay is expressed in terms of the delays associated with the various gates in the circuit. It is obtained by computing the total delay for every path from any input to the large block output (as was done in strategy no. 1 for the

longest path), summing all the total delays of the paths and dividing by the number of paths. The following notation is used. For an output rise transition associated with the j^{th} path beginning at the input node $\underline{Xk}$, the total propagation delay is denoted by $\underline{\tau}_{rkj}$ Similarly, for an output fall transition associated with the j^{th} path beginning at input node $\underline{Xk}$, the total propagation delay is denoted by $\underline{\tau}_{fkj}$. Let n denote the number of input signals and let n_k denote the number of distinct paths from the input node $\underline{Xk}$ to the output node $\underline{Z}$. The arithmetic mean delay for the output rise transition is denoted by $\overline{\underline{\tau}}_r$ and is given by

$$\overline{\underline{\tau}}_r = \sum_{k=1}^{n} \sum_{j=1}^{n_k} \underline{\tau}_{rkj} / \sum_{k=1}^{n} n_k \ . \tag{4.4}$$

Similarly, the arithmetic mean delay for the output fall transition, denoted by $\overline{\underline{\tau}}_f$, is given by

$$\overline{\underline{\tau}}_f = \sum_{k=1}^{n} \sum_{j=1}^{n_k} \underline{\tau}_{fkj} / \sum_{k=1}^{n} n_k . \tag{4.5}$$

Note that $\underline{\tau}_{rkj}$ and $\underline{\tau}_{fkj}$ are sums of rise and fall propagation delays associated with gates along the j^{th} path beginning at node $\underline{Xk}$. It follows that $\overline{\underline{\tau}}_r$ and $\overline{\underline{\tau}}_f$ are linear combinations of the rise and fall propagation delays of the various network gates. Assuming statistical independence among delays of different gates, the p.d.f.'s of $\overline{\underline{\tau}}_r$ and $\overline{\underline{\tau}}_f$ can be obtained

using the following rule. Let $\underline{Y} = \sum_{i=1}^{n} a_i \underline{xi}$, where the a_i s are constants and the $\underline{xi}$'s are statistically independent random variable with given p.d.f's $f_{\underline{xi}}(xi)$. Extending the result in [18, p.190] to several variables, the p.d.f of $\underline{Y}$ is given by

$$f_{\underline{Y}}(Y) = \frac{1}{\left|\prod_{i=1}^{n} a_i\right|} \quad f_{\underline{x1}}\left(\frac{Y}{a_1}\right)*\dots*f_{\underline{xn}}\left(\frac{Y}{a_n}\right) \tag{4.6}$$

The method is illustrated by the following example.

Example 4.3. Consider the circuit in Fig. 4.2. Note there are five distinct paths from the inputs to the output node $\underline{Z}$. It is desired to obtain in terms of the p.d.f's of the gate delays the p.d.f.'s of the mean path propagation delay for rise and fall transitions at the output $\underline{Z}(t)$.

The method used in Ex. 4.2 to determine the total delay for the longest path is used here to evaluate the total rise transition delay for each of the five paths. Recalling that $\underline{\tau}_{rkj}$ denotes the total rise transition delay for the j^{th} path beginning at input $\underline{Xk}$, it follows that

$$\underline{\tau}_{r11} = \underline{\tau}_{rA1} + \underline{\tau}_{r01}, \tag{4.7}$$

$$\underline{\tau}_{r21} = \underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02}, \tag{4.8}$$

$$\underline{\tau}_{r22} = \underline{\tau}_{rA2} + \underline{\tau}_{r02}, \tag{4.9}$$

$$\underline{\tau}_{r31} = \underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02}, \tag{4.10}$$

$$\underline{\tau}_{r32} = \underline{\tau}_{rA2} + \underline{\tau}_{r02} . \tag{4.11}$$

Summing Eqs.(4.7)-(4.11) and dividing by the number of paths as in Eq.(4.4), yield

$$\bar{\underline{\tau}}_r = (\underline{\tau}_{rA1} + \underline{\tau}_{r02} + \underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02} + \underline{\tau}_{rA2} + \underline{\tau}_{r02} + \underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02}$$

$$+ \underline{\tau}_{rA2} + \underline{\tau}_{r02})/5$$

$$= \frac{3}{5}\ \underline{\tau}_{rA1} + \frac{2}{5}\ \underline{\tau}_{rA2} + \frac{2}{5}\ \underline{\tau}_{r01} + \underline{\tau}_{r02}. \qquad (4.12)$$

Application of Eq. (4.6) to Eq. (4.12) results in

$$f_{\bar{\underline{\tau}}_r}(\tau) = \frac{125}{12}\ f_{\underline{\tau}rA}\left(\frac{5\tau}{3}\right) * f_{\underline{\tau}rA}\left(\frac{5\tau}{2}\right) * f_{\underline{\tau}r0}\left(\frac{5\tau}{2}\right) * f_{\underline{\tau}f0}(\tau). \qquad (4.13)$$

A similar derivation for $\bar{\underline{\tau}}_f$ results in

$$f_{\bar{\underline{\tau}}_f}(\tau) = \frac{125}{12}\ f_{\underline{\tau}fA}\left(\frac{5\tau}{3}\right) * f_{\underline{\tau}fA}\left(\frac{5\tau}{2}\right) * f_{\underline{\tau}f0}\left(\frac{5\tau}{2}\right) * f_{\underline{\tau}f0}(\tau). \qquad (4.14)$$

Although Eq.(4.13) and (4.14) are of the same form, the p.d.f's for $\bar{\underline{\tau}}_r$ and $\bar{\underline{\tau}}_f$ may differ significantly depending on the gate rise and fall delay. ∇

3. <u>Weighted mean path delay</u>. As in the case of mean path delay, all paths from every input to every output are considered. However, each path is now weighted according to the number of transitions likely

to occur in the path during the observation interval. Development of the weighting scheme is considered next. As in Eq. (2.79), let the derivative of the counting signals associated with the output of the ideal logic circuit be expressed by

$$\dot{\underline{z}}^{+}(t) = \sum_{k=1}^{n} \{\dot{\underline{Xk}}^{+}(t)\underline{Bk}^{+}(t) + \dot{\underline{Xk}}^{-}(t)\underline{Bk}^{-}(t)\}$$

and

$$\dot{\underline{z}}^{-}(t) = \sum_{k=1}^{n} \{\dot{\underline{Xk}}^{+}(t)\underline{Bk}^{-}(t) + \dot{\underline{Xk}}^{-}(t)\underline{Bk}^{+}(t)\} \ . \tag{4.15}$$

The coefficients $\underline{Bk}^{+}(t)$ and $\underline{Bk}^{-}(t)$ are assumed to be in their arithmetic canonical form as presented in Eqs.(2.72),(2.74), and (2.75). Recall that $\underline{Pk}(m)$ contributes to either $\underline{Bk}^{+}$ or $\underline{Bk}^{-}$, but not both, and consists of a distinct combination of input variables or their complements.

When $\underline{Pk}(m) = 1$, either $\underline{Bk}^{+}(t)$ or $\underline{Bk}^{-}(t)$ equals unity, but not both. From Eq.(4.15) it follows that a transition in $\underline{Xk}(t)$ will produce a transition in $\underline{z}(t)$. Consequently, when $Pk(m) = 1$, the gates in the network are enabled such that a transition in $\underline{Xk}(t)$ propagates from the node $\underline{Xk}$ to the output node $\underline{z}$. Several such paths may be activated simultaneously when $\underline{Pk}(m) = 1$. Denote the collection of such paths by π_{km}. With reference to Eqs.(4.15) it is clear that a rise transition occurs at the output of each path in π_{km} provided either

1) $\underline{Pk}(m) = 1$ yields $\underline{Bk}^{+}(t) = 1$ and $\underline{Xk}(t)$ has a rise transition

or

2) $\underline{Pk}(m) = 1$ yields $Bk^{-}(t) = 1$ and $\underline{Xk}(t)$ has a fall transition.

Similarly, a fall transition occurs at the output of each path in π_{km}

provided either

1) $\underline{Pk}(m) = 1$ yields $\underline{Bk}^{+}(t) = 1$ and $\underline{Xk}(t)$ has a fall transition

or

2) $\underline{Pk}(m) = 1$ yields $\underline{Bk}^{-}(t) = 1$ and $\underline{Xk}(t)$ has a rise transition.

Denote by A_{km} the event that the collection of paths π_{km} are activated such that transitions in $\underline{Xk}(t)$ can propagate to the output $\underline{z}(t)$. This is equivalent to the event that $\underline{Pk}(m) = 1$. Consequently, the probability that the collection of paths π_{km} are activated is given by

$$\Pr\{A_{km}\} = \Pr\{\underline{Pk}(m) = 1\} = E[\underline{Pk}(m)]. \tag{4.16}$$

Attention is now devoted to determining the expected rate of transitions in $\underline{Xk}(t)$. To be consistent with Eq.(3.40) and (3.41) it is assumed, without loss of generality, that

$$\underline{Xk}^{-}(0) = 0$$

$$\underline{Xk}^{+}(0) = \underline{Xk}(0). \tag{4.17}$$

Given the sample function $Xk(t)$, it follows for $t_A > 0$ that $Xk^{+}(t_A)$ equals the number of rise transitions in the interval $[0,t_A]$ while $Xk^{-}(t_A)$ equals the number of fall transitions in $[0,t_A]$. The rates of rise and fall transitions in $Xk(t)$ at some time $t \in [0,t_A]$ are given by $\dot{X}k^{+}(t)$ and $\dot{X}k^{-}(t)$, respectively. Taking the expected values over the ensembles of $\underline{\dot{X}k}^{+}(t)$ and $\underline{\dot{X}k}^{-}(t)$ and interchanging the order of expectation and differentiation, as in Eq. (2.81), it follows that $\dot{E}[\underline{Xk}^{+}(t)]$ and $\dot{E}[\underline{Xk}^{-}(t)]$ are the expected rates of rise and fall transitions in $\underline{Xk}(t)$ respectively.

At any time t the expected rate of input rise and fall transitions which propagate through the collection of paths π_{km} are given by $\dot{E}[\underline{Xk}^{+}(t)]E[\underline{Pk}(m)]$ and $\dot{E}[\underline{Xk}^{-}(t)]E[\underline{Pk}(m)]$, respectively. The output transition being a rise or fall depends upon whether $\underline{Pk}(m) = 1$ causes $\underline{Bk}^{+}(t) = 1$ or $\underline{Bk}^{-}(t) = 1$. Let the time average of a function $h(t)$ over an interval (t_o, t_f) be given by

$$< h(t) > = \frac{1}{t_f - t_o} \int_{t_o}^{t_f} h(t)dt. \tag{4.18}$$

Taking time averages over the observation interval, denote the time averages of the expected rate of output rise and fall transitions due to transitions propagated through the paths π_{km} by a_{rkm} and a_{fkm}, respectively. It follows that

$$a_{rkm} = \begin{cases} < \dot{E}[\underline{Xk}^{+}(t)]E[\underline{Pk}(m)] > , & \underline{Pk}(m) = 1 \Rightarrow \underline{Bk}^{+} = 1 \\ < \dot{E}[\underline{Xk}^{-}(t)]E[\underline{Pk}(m)] > , & Pk(m) = 1 \Rightarrow \underline{Bk}^{-} = 1. \end{cases} \tag{4.19}$$

and

$$a_{fkm} = \begin{cases} < \dot{E}[\underline{Xk}^{-}(t)]E[\underline{Pk}(m)] > , & \underline{Pk}(m) = 1 \Rightarrow \underline{Bk}^{+} = 1. \\ < \dot{E}[\underline{Xk}^{+}(t)]E[\underline{Pk}(m)] > , & \underline{Pk}(m) = 1 \Rightarrow \underline{Bk}^{-} = 1. \end{cases} \tag{4.20}$$

a_{rkm} and a_{fkm} are, in a sense, measures of usage for the paths π_{km}.

To determine the rise and fall delays associated with each of the paths in π_{km}, it is necessary to identify the various gates included in each path. This is done by first assigning to the inputs $\underline{Xj}$, $j=1,\ldots,n;\ j \neq k$,

those values which result in $\underline{Pk}(m)=1$. By tracing each path from the input node $\underline{Xk}$ to the output node $\underline{z}$, it is noted which gates experience transitions due to a transition in $\underline{Xk}(t)$. Paths for which transitions do not propagate completely to node $\underline{z}$ are ignored. Denote by μ_{rkm} and μ_{fkm} the number of paths, respectively, which produce rise and fall transitions in $\underline{z}(t)$ due to a transition in $\underline{Xk}(t)$. Using the same method as in strategies 1 and 2, a total delay is determined for each one of the $(\mu_{rkm} + \mu_{fkm})$ paths. Denote by $\underline{\tau}_{rkm}$ and $\underline{\tau}_{fkm}$ the arithmetic mean delays for the μ_{rkm} and μ_{fkm} paths, respectively.

Let γ_k denote the number of nonzero coefficients $\underline{Bk(m)}$ in the summation for $\underline{Bk}$ given by Eq.(2.72). γ_k equals the number of distinct collections of paths π_{km} from the input node $\underline{Xk}$ to the output node $\underline{z}$. Taking into consideration each of the inputs to the large logic block and performing a weighted arithmetic mean over the $\sum_{k=1}^{n} \gamma_k$ path delays, the weighted mean path delays for the output rise and fall transitions in $\underline{z}(t)$, $<\underline{\tau}_r>$ and $<\underline{\tau}_f>$, are defined to be

$$< \underline{\tau}_r > = \sum_{k=1}^{n} \sum_{m=1}^{\gamma_k} a_{rkm}\, \underline{\tau}_{rkm} \Big/ \sum_{k=1}^{n} \sum_{m=1}^{\gamma_k} a_{rkm} \tag{4.21}$$

and

$$< \underline{\tau}_f > = \sum_{k=1}^{n} \sum_{m=1}^{\gamma_k} a_{fkm}\, \underline{\tau}_{rkm} \Big/ \sum_{k=1}^{n} \sum_{m=1}^{\gamma_k} a_{fkm}. \tag{4.22}$$

Because of the weights employed in Eqs. (4.21) and (4.22), $<\underline{\tau}_r>$ and $<\underline{\tau}_f>$ reflect the usage of the paths π_{km}.

As in Eqs.(4.4) and (4.5), $<\underline{\tau}_r>$ and $<\underline{\tau}_f>$ are linear combinations of rise and fall propagation delays associated with individual gates. Assuming statistical independence among delays of different gates, the p.d.f.'s $f_{<\underline{\tau}r>}(\tau)$ and $f_{<\underline{\tau}f>}(\tau)$ can be obtained by application of Eq.(4.6). The previous discussion is illustrated by the following example.

<u>Example 4.4</u>. Consider again the 2 out of 3 majority logic circuit shown in Fig. 4.2. The switching operation performed by this circuit is $Z = X1 \cdot X2 \vee X1 \cdot X3 \vee X2 \cdot X3$. It is desired to obtain expressions for $f_{<\underline{\tau}r>}(\tau)$ and $f_{<\underline{\tau}f>}(\tau)$ when this circuit is considered as a single large logic block.

The large logic block model for the circuit in Fig. 4.2 is depicted in Fig. 4.3. The output of the ideal logic circuit is given by

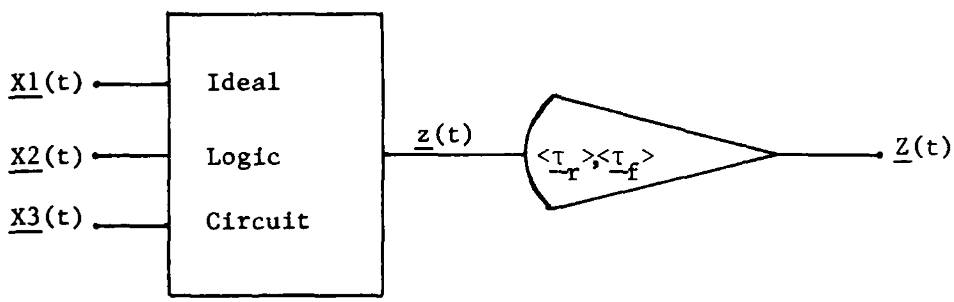


Fig. 4.3. Large logic block model for circuit in Fig. 4.2

$$\underline{z}(t) = \underline{X1}(t) \cdot \underline{X2}(t) \vee \underline{X1}(t) \cdot \underline{X3}(t) \vee \underline{X2}(t) \cdot \underline{X3}(t). \tag{4.23}$$

First, the derivatives $\dot{\underline{z}}^+(t)$ and $\dot{\underline{z}}^-(t)$ are obtained in their arithmetic canonical form. An arithmetic expression for $\underline{z}(t)$ in Eq.(4.23) is given by

$$\underline{z}(t) = \underline{X1}(t)\underline{X2}(t) + \underline{X1}(t)\underline{X2}'(t)\underline{X3}(t) + \underline{X1}'(t)\underline{X2}(t)\underline{X3}(t). \tag{4.24}$$

Differentiation of Eq. (4.24) yields

$$\dot{\underline{z}}(t) = \dot{\underline{X1}}(t)\underline{X2}(t) + \dot{\underline{X2}}(t)\underline{X1}(t) + \dot{\underline{X1}}(t)\underline{X2}'(t)\underline{X3}(t) + \dot{\underline{X2}}'(t)\underline{X1}(t)\underline{X3}(t)$$

$$+ \dot{\underline{X3}}(t)\underline{X1}(t)\underline{X2}'(t) + \dot{\underline{X1}}'(t)\underline{X2}(t)\underline{X3}(t) + \dot{\underline{X2}}(t)\underline{X1}'(t)\underline{X3}(t)$$

$$+ \dot{\underline{X3}}(t)\,\underline{X1}'(t)\underline{X2}(t). \tag{4.25}$$

Applying Eq.(2.62) to Eq.(4.25) and collecting terms result in

$$\dot{\underline{z}}(t) = \dot{\underline{X1}}(t)\{\underline{X2}(t) + \underline{X2}'(t)\underline{X3}(t) - \underline{X2}(t)\underline{X3}(t)\} + \dot{X2}(t)\{\underline{X1}(t) - \underline{X1}(t)\underline{X3}(t)$$

$$+ \underline{X1}'(t)\underline{X3}(t)\} + \dot{\underline{X3}}(t)\{\underline{X1}(t)\underline{X2}'(t) + \underline{X1}'(t)\underline{X2}(t)\} \tag{4.26}$$

With reference to Eq.(2.63),

$$\underline{B1}(t) = \underline{X2}(t) + \underline{X2}'(t)\,\underline{X3}(t) - \underline{X2}(t)\underline{X3}(t)$$

$$\underline{B2}(t) = \underline{X1}(t) - \underline{X1}(t)X3(t) + \underline{X1}'(t)\underline{X3}(t)$$

$$\underline{B3}(t) = \underline{X1}(t)\,\underline{X2}'(t) + \underline{X1}'(t)\underline{X2}(t). \tag{4.27}$$

As discussed on page 43, the arithmetic canonical forms for $\underline{Bk}(t)$, k=1,2,3, are obtained from Table 4.1 shown below:

TABLE 4.1

$\underline{B1}$

m	X2	X3	B1(m)	P1(m)
1	0	0	0	$(\underline{X2}')(\underline{X3}')$
2	0	1	1	$(\underline{X2}')(\underline{X3})$
3	1	0	1	$(\underline{X2})(\underline{X3}')$
4	1	1	0	$(\underline{X2})(\underline{X3})$

$\underline{B2}$

m	X1	X3	B2(m)	P2(m)
1	0	0	0	$(\underline{X1}')(\underline{X3}')$
2	0	1	1	$(\underline{X1}')(\underline{X3})$
3	1	0	1	$(\underline{X1})(\underline{X3}')$
4	1	1	0	$(\underline{X1})(\underline{X3})$

TABLE 4.1 (continued)

$\underline{B3}$

m	X1	X2	B3(m)	P3(m)
1	0	0	0	$(\underline{X1}')(\underline{X2}')$
2	0	1	1	$(\underline{X1}')(\underline{X2})$
3	1	0	1	$(\underline{X1})(\underline{X2}')$
4	1	1	0	$(\underline{X1})(\underline{X2})$

With reference to Table 4.1 the arithmetic canonical forms for $\underline{Bk}(t)$, $k = 1,2,3$, are

$$\underline{B1}(t) = \underline{X2}'(t)\underline{X3}(t) + \underline{X2}(t)\underline{X3}'(t)$$

$$\underline{B2}(t) = \underline{X1}'(t)\underline{X3}(t) + \underline{X1}(t)\underline{X3}'(t)$$

$$\underline{B3}(t) = \underline{X1}'(t)\underline{X2}(t) + \underline{X1}(t)\underline{X2}'(t). \tag{4.28}$$

From Eqs.(2.74) and (2.75) it follows that

$$\begin{aligned}
\underline{B1}^{+}(t) &= \underline{X2}'(t)\underline{X3}(t) + \underline{X2}(t)\underline{X3}'(t) \\
&= \underline{P1}(2) + \underline{P1}(3), \\
\underline{B2}^{+}(t) &= \underline{X1}'(t)\underline{X3}(t) + \underline{X1}(t)\underline{X3}'(t) \\
&= \underline{P2}(2) + \underline{P2}(3), \\
\underline{B3}^{+}(t) &= \underline{X1}'(t)\underline{X2}(t) + \underline{X1}(t)\,\underline{X2}'(t) \\
&= \underline{P3}(2) + \underline{P3}(3), \\
\underline{B1}^{-}(t) &= \underline{B2}^{-}(t) = \underline{B3}^{-}(t) = 0.
\end{aligned} \tag{4.29}$$

The substitution of Eqs.(4.29) into (4.15) results in

$$\begin{aligned}
\dot{\underline{z}}^{+}(t) &= \sum_{k=1}^{3} \dot{\underline{Xk}}^{+}(t)[\underline{Pk}(2) + \underline{Pk}(3)] \\
\dot{\underline{z}}^{-}(t) &= \sum_{k=1}^{3} \dot{\underline{Xk}}^{-}(t)[\underline{Pk}(2) + \underline{Pk}(3)].
\end{aligned} \tag{4.30}$$

The conditions under which $\underline{Pk}(m) = 1$ for $k = 1,2,3$ and $m = 2,3$ are tabulated in Table 4.2.

TABLE 4.2

	P1(2)	P1(3)	P2(2)	P2(3)	P3(2)	P3(3)
	(X2')(X3)	(X2)(X3')	(X1')(X3)	(X1)(X3')	(X1')(X2)	(X1)(X2')
$\underline{X1}$	--	-	0	1	0	1
$\underline{X2}$	0	1	-	-	1	0
$\underline{X3}$	1	0	1	0	-	-

Referring to Eqs.(4.19) and (4.20) and utilizing statistical independence, the expressions for the weights a_{rkm} and a_{fkm} are given by

$$a_{r12} = \langle \dot{E}[\underline{X1}^+(t)]\ E[\underline{X2}'(t)]\ E[\underline{X3}(t)] \rangle$$

$$a_{r13} = \langle \dot{E}[\underline{X1}^+(t)]\ E[\underline{X2}(t)]\ E[\underline{X3}'(t)] \rangle$$

$$a_{r22} = \langle \dot{E}[\underline{X2}^+(t)]\ E[\underline{X1}'(t)]E[\underline{X3}(t)] \rangle$$

$$a_{r23} = \langle \dot{E}[\underline{X2}^+(t)]\ E[\underline{X1}(t)]\ E[\underline{X3}'(t)] \rangle$$

$$a_{r32} \quad \langle \dot{E}[\underline{X3}^+(t)]\ E[\underline{X1}'(t)]E[\underline{X2}(t)] \rangle$$

$$a_{r33} = \langle \dot{E}[\underline{X3}^+(t)]\ E[\underline{X1}(t)]E[\underline{X2}'(t)] \rangle \tag{4.31}$$

and

$$a_{f12} = \langle \dot{E}[\underline{X1}^-(t)]E[\underline{X2}'(t)]E[\underline{X3}(t)] \rangle$$

$$a_{f13} = \langle \dot{E}[\underline{X1}^-(t)]E[\underline{X2}(t)]E[\underline{X3}'(t)] \rangle$$

$$a_{f22} = \langle \dot{E}[\underline{X2}^-(t)]E[\underline{X1}'(t)]E[\underline{X3}(t)] \rangle$$

$$a_{f23} = \langle \dot{E}[\underline{X2}^-(t)]E[\underline{X1}(t)]E[\underline{X3}'(t)] \rangle$$

$$a_{f32} = < \dot{E}[\underline{X3}^-(t)]E[\underline{X1}'(t)]E[\underline{X2}(t)] >$$

$$a_{f33} = < \dot{E}[\underline{X3}^-(t)]E[\underline{X1}(t)]E[\underline{X2}'(t)] > . \qquad (4.32)$$

The next step is to identify the paths π_{km} and determine the total rise and fall propagation delays associated with each path. The gates in each path, μ_{rkm}, $\underline{\tau}_{rkm}$, μ_{fkm}, and $\underline{\tau}_{fkm}$ are listed in Table 4.3. To illustrate how entries in the Table are obtained, focus attention

TABLE 4.3

k	m	Gates in Path π_{km}	μ_{rkm}	$\underline{\tau}_{rkm}$	μ_{fkm}	$\underline{\tau}_{fkm}$
1	2	A1,02	1	$\underline{\tau}_{rA1} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{fA1} + \underline{\tau}_{f02}$
1	3	A1,02	1	$\underline{\tau}_{rA1} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{fA1} + \underline{\tau}_{f02}$
2	2	A2,02	1	$\underline{\tau}_{rA2} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{fA2} + \underline{\tau}_{f02}$
2	3	01,A1,02	1	$\underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{f01} + \underline{\tau}_{fA1} + \underline{\tau}_{f02}$
3	2	A2,02	1	$\underline{\tau}_{rA2} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{fA2} + \underline{\tau}_{f02}$
3	3	01,A1,02	1	$\underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02}$	1	$\underline{\tau}_{f01} + \underline{\tau}_{fA1} + \underline{\tau}_{f02}$

on the cases k = 2, m = 2 and k = 2, m = 3. For k = 2, m = 2, $\underline{P2}(2) = (\underline{X1}')(\underline{X3}) = 1$ provided $\underline{X1} = 0$, $\underline{X3} = 1$. For these values of $\underline{X1}(t)$ and $\underline{X3}(t)$, the gate A1 is inhibited while the gates A2 and 02 are enabled. Therefore, a transition in $\underline{X2}(t)$ propagates through the path π_{22} consisting of gates A2 and 02. Observe that rise and fall transitions in $\underline{X2}(t)$ produce rise and fall transitions, respectively, at the outputs of gates A2 and 02. For k = 2, m = 3, $\underline{P2}(3) = (\underline{X1})(\underline{X3}') = 1$ provided

$\underline{X1}=1$, $\underline{X3}=0$. For these values of $\underline{X1}(t)$ and $\underline{X3}(t)$, the gate A2 is inhibited while gates 01, A1, and 02 are enabled. Therefore, a transition in $\underline{X2}(t)$ propagates through the path π_{23} consisting of gates 01, A1, and 02. As before, rise and fall transitions in $\underline{X2}(t)$ produce rise and fall transitions, respectively, at the outputs of gates 01, A1, and 02.

Finally, the results from Eqs.(4.31), (4.32), and Table 4.3 are substituted into Eqs.(4.21) and (4.22) to obtain expressions for $<\underline{\tau}_r>$ and $<\underline{\tau}_f>$. Eq.(4.6) is then employed to obtain the output p.d.f's $\tau_{<\underline{\tau}r>}(\tau)$ and $f_{<\underline{\tau}f>}(\tau)$.

4. Assigned Gaussian delay. The assigned Gaussian delay strategy was motivated by Magnhagen and Flisberg[12] who assigned Gaussian p.d.f.' to each gate delay in their simulation program called DIGSIM. However, in the strategy proposed here, the large logic block is modeled by simply assigning rise and fall Gaussian delays at each block output. The Gaussi parameters are obtained as discussed below.

The mean and variance for each Gaussian p.d.f. are determined from the minimum and maximum delays experienced by transitions propagating fro any input to the output under consideration. Denote these rise and fall delays by τ_{rmin}, τ_{rmax}, τ_{fmin}, and τ_{fmax}. It is assumed that minimum and maximum delays are known for each gate within the large logic block. τ_{rm} is determined by 1) identifying the paths with the minimum number of gate 2) evaluating the minimum output rise propagation delay for each path by

summing the appropriate minimum delay for each gate along the path, and 3) assigning to τ_{rmin} the minimum of the sums evaluated in step (2). τ_{fmin} is determined in like manner. τ_{rmax} is determined by 1) identifying the paths with the maximum number of gates, 2) evaluating the maximum output rise propagation delay for each path by summing the appropriate maximum delay for each gate along the path, and 3) assigning to τ_{rmax} the maximum of the sums evaluated in step (2). A similar procedure is used to obtain τ_{fmax}.

Let the random variables for the rise and fall delays assigned to the output be denoted by $\underline{\tau}_{rG}$ and $\underline{\tau}_{fG}$, respectively. By definition, the means are selected to be the averages of the maximum and minimum delays. Hence,

$$E[\underline{\tau}_{rG}] = \frac{1}{2}(\tau_{rmax} + \tau_{rmin}) \tag{4.33}$$

$$E[\underline{\tau}_{fG}] = \frac{1}{2}(\tau_{fmax} + \tau_{fmin}). \tag{4.34}$$

The variances are determined by assigning the 3σ points of the Gaussian p.d.f. to the maximum and minimum delays. Therefore,

$$\begin{aligned} E[\underline{\tau}_{rG}] - 3\sigma_{\underline{\tau}rG} &= \tau_{rmin} \\ E[\underline{\tau}_{rG}] + 3\sigma_{\underline{\tau}rG} &= \tau_{rmax} \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} E[\underline{\tau}_{fG}] - 3\sigma_{\underline{\tau}fG} &= \tau_{fmin} \\ E[\underline{\tau}_{fG}] + 3\sigma_{\underline{\tau}fG} &= \tau_{fmax}. \end{aligned} \tag{4.36}$$

Solution of Eqs. (4.35) and (4.36) for the standard deviations result in

$$\sigma_{\underline{\tau}rG} = \frac{1}{6}\,(\tau_{rmax} - \tau_{rmin}) \tag{4.37}$$

$$\sigma_{\underline{\tau}fG} = \frac{1}{6}\,(\tau_{fmax} - \tau_{fmin}). \tag{4.38}$$

The approximated Gaussian p.d.f.'s for the output delays are given by

$$f_{\underline{\tau}rG}(\tau_r) = \frac{6}{\sqrt{2\pi}\,(\tau_{rmax}-\tau_{rmin})}\, e^{\left(-\frac{18[\tau_r - \frac{1}{2}(\tau_{rmax}+\tau_{rmin})]^2}{(\tau_{rmax} - \tau_{rmin})^2}\right)} \tag{4.39}$$

$$f_{\underline{\tau}fG}(\tau_f) = \frac{6}{\sqrt{2\pi}\,(\tau_{fmax}-\tau_{fmin})}\, e^{\left(-\frac{18[\tau_f - \frac{1}{2}(\tau_{fmax}+\tau_{fmin})]^2}{(\tau_{fmax} - \tau_{fmin})^2}\right)} \tag{4.40}$$

The strategy is illustrated by the following example.

Example 4.5. Consider again the circuit in Fig. 4.2. Suppose the minimum and maximum values for the rise and fall propagation delays associated with the AND and OR gates are known to be

$$0.1 < \underline{\tau}_{rA} < 0.6 \qquad 0.2 < \underline{\tau}_{fA} < 0.5 \tag{4.41}$$

$$0.2 < \underline{\tau}_{rO} < 0.9 \qquad 0.3 < \underline{\tau}_{fO} < 1.0 \tag{4.42}$$

where it is assumed that like gates have the same minimum and maximum delays. It is desired to assign Gaussian p.d.f.'s to the output delays

when the entire circuit of Fig. 4.2 is modeled as a large logic block.

For the circuit of Fig. 4.2, the minimum number of gates in a path equals two. There are three paths with two gates: 1) a path from node X1 to node Z via gates A1 and 02, 2) a path from node X2 to node Z via gates A2 and 02, and 3) a path from node X3 to node Z via gates A2 and 02. Observe that rise and fall transitions propagate in the same manner through each path. Since all three paths contain one AND gate and one OR gate and like gates have the same minimum delays, only one of the three paths need be analyzed.

It follows that

$$\tau_{rmin} = \min\{\underline{\tau}_{rA}\} + \min\{\underline{\tau}_{r0}\}$$

$$= 0.1 + 0.2 = 0.3 \ , \qquad (4.43)$$

$$\tau_{fmin} = \min\{\underline{\tau}_{fA}\} + \min\{\underline{\tau}_{f0}\}$$

$$= 0.2 + 0.3 = 0.5 \qquad (4.44)$$

The maximum number of gates contained in a path equals three. There are two paths with three gates: 1) a path from node X2 to node Z via gates 01, A1 and 02, and 2) a path from node X3 to node Z via gates 01, A1 and 02. As before, rise and fall transitions propagate in the same manner through each path. Since both paths contain the same gates, it follows that

$$\tau_{rmax} = \max\{\underline{\tau}_{r0}\} + \max\{\underline{\tau}_{rA}\} + \max\{\underline{\tau}_{r0}\}$$

$$= 0.9 + 0.6 + 0.9 = 2.4, \qquad (4.45)$$

$$\tau_{fmax} = \max\{\underline{\tau}_{f0}\} + \max\{\underline{\tau}_{fA}\} + \max\{\underline{\tau}_{f0}\}$$

$$= 1.0 + 0.5 + 1.0 = 2.5. \qquad (4.46)$$

Substitution of τ_{rmin} and τ_{rmax} in Eq.(4.39) and τ_{fmin} and τ_{fmax} into Eq. (4.40) result in

$$f_{\underline{\tau}rG}(\tau_r) = \frac{6}{2.1\sqrt{2\pi}}\, e^{-\frac{18[\tau_r-1.35]^2}{(2.1)^2}}$$

$$= 1.14\, e^{-4.08(\tau_r-1.35)^2} \qquad (4.47)$$

$$f_{\underline{\tau}fG}(\tau_f) = \frac{6}{2\sqrt{2\pi}}\, e^{-\frac{18[\tau_f-1.5]^2}{2^2}}$$

$$= 1.20\, e^{-4.5(\tau_f-1.5)^2} \qquad (4.48)$$

The p.d.f.'s for $\underline{\tau}_{rG}$ and $\underline{\tau}_{fG}$, are sketched in Fig. 4.4.

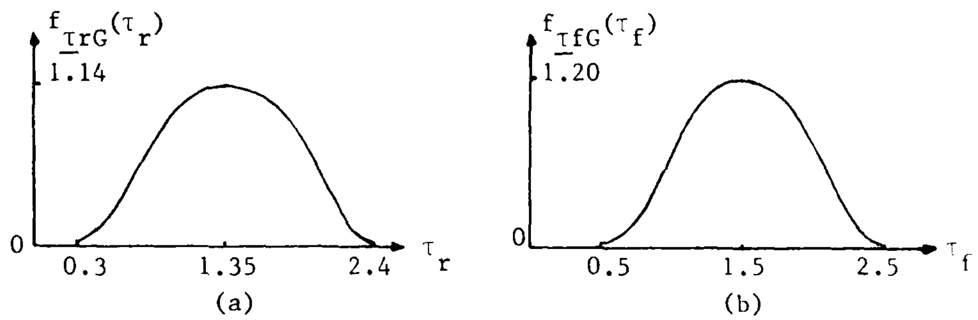


Fig. 4.4 Sketches of the p.d.f's for a) $\underline{\tau}_{rG}$ and b) $\underline{\tau}_{fG}$.

As a final point, it is noted that computer simulation programs, such as LDPS by Al-Hussein and Dutton [14], may be employed to obtain τ_{rmin}, τ_{rmax}, τ_{fmin}, and τ_{fmax}.

4.3 Comparison of Output Delay Characterizations

The four strategies discussed in Sec. 4.2 are compared in this section. Specific p.d.f.'s are assigned to the individual gates in the circuit of Fig. 4.2. The circuit is modeled as a large logic block using the simple output delay model shown in Fig. 3.10. For each strategy, the p.d.f.'s of $\underline{\tau}_r$ and $\underline{\tau}_f$ are determined and $E[\underline{Z}(t)]$ is evaluated. In addition, a Monte Carlo simulation is performed to provide a reference waveform for $E[\underline{Z}(t)]$.

An exact analysis of the circuit was not performed because of its complexity (see App.B). Several considerations entered into the design of the Monte Carlo simulation. The first step in each run was to assign numerical values to the gate rise and fall propagation delays. Random number generators which reflected the assigned delay p.d.f.'s were used for this purpose. A new set of gate delays was randomly and independently selected for each run and the output signal obtained. Let the output of the i^{th} run be denoted by $Z_i(t)$. Also, let N denote the total number of runs. The reference waveform derived from the Monte Carlo simulation was defined to be

$$Z_M(t) = \frac{1}{N} \sum_{i=1}^{N} Z_i(t). \tag{4.49}$$

The accuracy of the Monte Carlo simulation is now discussed. Consider a particular time instant, t_1, in the observation time interval. Denote

by P_1 the probability that $\underline{Z}(t_1) = 1$. Specifically,

$$P_1 = P_r\{\underline{Z}(t_1) = 1\} = E[\underline{Z}(t_1)]. \tag{4.50}$$

Since $\underline{Z}(t_1)$ is a binary random variable with values 0 or 1, the Monte Carlo simulation for $t = t_1$ can be interpreted as repeated Bernoulli trials with probability P_1[18, Sec.3-2] assuming the runs of the Monte Carlo simulation to be statistically independent. Let N_1 denote the number of runs in which $\underline{Z}(t_1) = 1$. Because $\underline{Z}(t_1)$ is a 0,1 binary random variable and $Z_i(t_1)$ denotes the i^{th} sample, $N_1 = \sum_{i=1}^{N} Z_i(t_1)$. By the law of large numbers [18],

$$\lim_{N\to\infty} \frac{N_1}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Z_i(t_1) = P_1 . \tag{4.51}$$

Comparison of Eqs. (4.49) and (4.51) reveals that $Z_M(t_1)$ is a consistent estimator of P_1. The second order moment of $\underline{Z}(t_1)$ is given by

$$E[\underline{Z}^2(t_1)] = (1)^2 P_r\{\underline{Z}(t_1) = 1\} + (0)^2 P_r\{\underline{Z}(t_1) = 0\} = P_1. \tag{4.52}$$

This result can also be obtained using the idempotency property of the 0,1 binary random variable. Since $\underline{Z}^2(t_1) = \underline{Z}(t_1)$, it follows that

$$E[\underline{Z}^2(t_1)] = E[\underline{Z}(t_1)] = P_1 . \tag{4.53}$$

The variance, $\sigma^2_{\underline{Z}(t_1)}$, is given by

$$\sigma^2_{\underline{Z}(t_1)} = E[\underline{Z}^2(t_1)] - \{E[Z(t_1)]\}^2 = P_1 - P_1{}^2 = P_1(1 - P_1). \tag{4.54}$$

When the Monte Carlo simulation is repeated many times, the waveforms in Eq.(4.49) can be viewed as random processes. Then

$$\underline{Z}_M(t_1) = \frac{1}{N} \sum_{i=1}^{N} \underline{Z}_i(t_1) . \tag{4.55}$$

The expected value of $\underline{Z}_M$ and its variance, respectively are readily shown to be [18, p.246]

$$E[\underline{Z}_M(t_1)] = P_1 \tag{4.56}$$

$$\sigma^2_{\underline{Z}M(t_1)} = \frac{P_1(1-P_1)}{N} . \tag{4.57}$$

Assuming N to be large and applying the central limit theorem to Eq.(4.55), $\underline{Z}_M(t_1)$ can be approximated as a Guassian random variable with mean and variance given by Eqs.(4.56) and (4.57).

For a Guassian p.d.f, 99% of the area is located in the symmetric interval $I_{0.99}$, extending 2.6 standard deviations on each side of the expected value. It follows that

$$P_r \ \{E[\underline{Z}(t_1)] - 2.6\ \sigma_{\underline{Z}M(t_1)} \le \underline{Z}_M(t_1) \le E[\underline{Z}(t_1)] + 2.6\ \sigma_{\underline{Z}M(t_1)}\} = 0.99 \tag{4.58}$$

Therefore, with probability 0.99, each outcome of the Monte Carlo simulation falls in the interval $I_{0.99}$ Note that the center of the interval is located at the true $E[\underline{Z}(t_1)]$. The width of the interval is given by 2ε where

$$\varepsilon = 2.6\sqrt{\frac{P_1(1-P_1)}{N}} \tag{4.59}$$

For $P_1 = 0.5$, $P_1(1-P_1)$ equals its maximum value of 0.25. With N = 1000, the interval half width is bounded by

$$\varepsilon \leq 2.6\sqrt{\frac{0.25}{1000}} = 0.041. \tag{4.60}$$

To determine a reference waveform, a single Monte Carlo simualation was performed with N = 1000. Eq.(4.60) can be viewed as an upper bound on the simulation error. Values of ε and $\varepsilon/E[\underline{Z}(t_1)]$ are tabulated in Table 4.4 for various values of $P_1 = E[\underline{Z}(t_1)]$. It is seen that the Monte Carlo simulation yields a reasonable estimation of $E[\underline{Z}(t)]$.

TABLE 4.4

P_1	10^{-3}	10^{-2}	0.1	0.3	0.5	0.7	0.9	0.99	0.999
ε	.0026	.0082	.0247	.0377	.0411	.0377	.0247	.0082	.0026
$\frac{\varepsilon}{E[\underline{Z}(t_1)]}$	2.6	.82	.247	.1256	.0822	.0539	.0247	.0082	.0026

To reflect the actual behavior of a physical circuit, an additional effect was taken into consideration in the Monte Carlo simulation. The simulation program ignored all pulses at a gate input whose durations δ_i were less than the rise propagation delay, τ_r, of the gate and all gaps whose duration γ_j were less than the fall propagation delay, τ_f, of the gate. This effect

is referred to as blanking. Indeed, in physical gates, τ_r and τ_f typically approximate the response times of the circuit. Therefore, the circuit does not respond to pulses and gaps of shorter durations. The effect of blanking in the probabilistic analysis was ignored because it was not possible to develop an analytical procedure for determining the statistics of the pulse and gap durations as the signals propagated through the various gates in the circuit. The results of this chapter indicate that blanking in the large block can be ignored when the pulse and gap durations at the output of the ideal logic circuit are longer than the maximum delays assigned to the discriminating delay element.

As mentioned at the outset of this section, the 2 out of 3 majority logic circuit in Fig. 4.2 was used as a vehicle for comparing the four strategies. For convenience, the circuit is shown again in Fig. 4.5. The p.d.f's of the delays assumed for the gates of this circuit

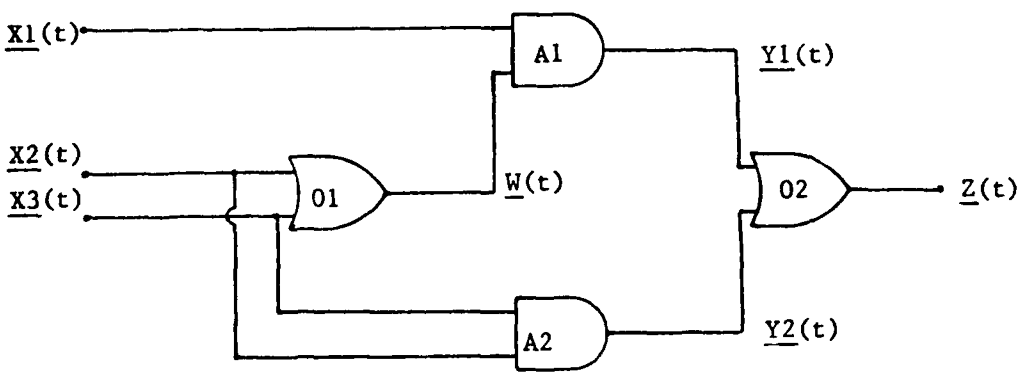


Fig. 4.5. 2 our of 3 majority logic circuit.

are sketched in Fig. 4.6, and the deterministic input signals, X1(t), X2(t) and X3(t) are depicted in Fig. 4.7.

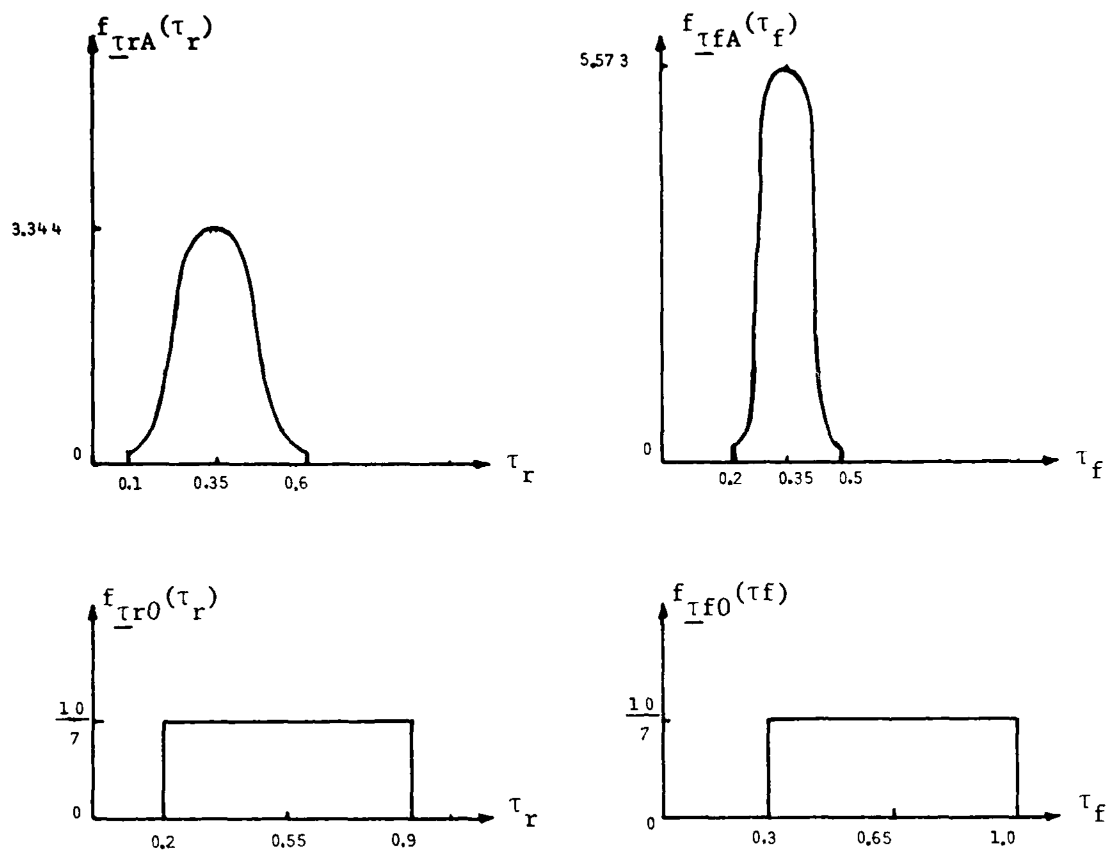


Fig. 4.6. Sketch of p.d.f's of assumed delays for gates in Fig. 4.5.

The circuit is intended to perform majority logic on its three inputs. In other words, the output should take on the value 1 if two or more inputs equal 1 and the value 0 if two or more inputs are zero. Clearly, in the large logic block model of the circuit in Fig. 4.5, the ideal logic portion will perform majority logic. z(t), the output of the ideal logic portion is also sketched in Fig. 4.7. Note that pulse Az is due to the overlapping of pulse $X1_1$ of X1(t) and pulse $X3_1$ of X3(t).

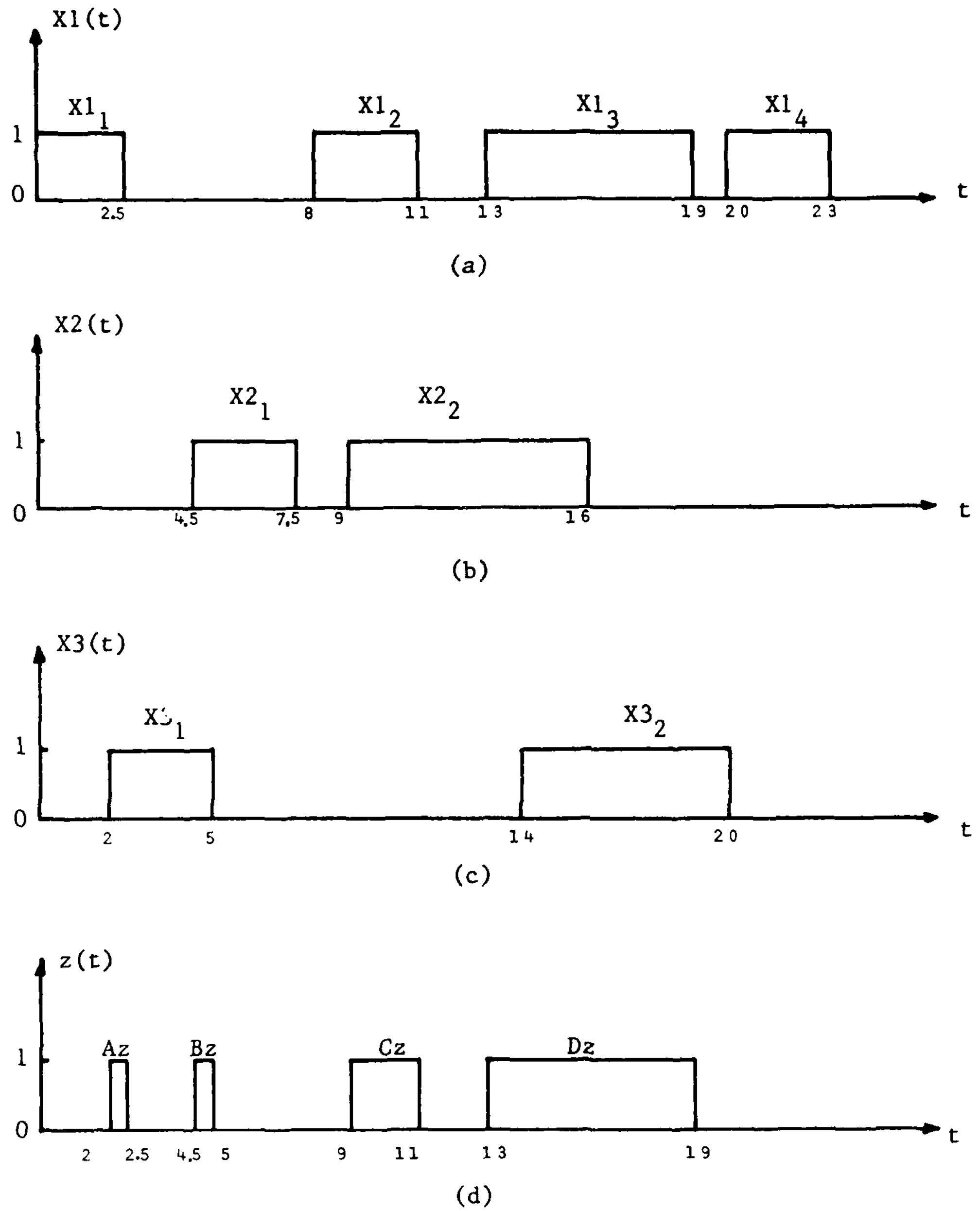


Fig. 4.7. Input signals (a) X1(t), (b) X2(t), (c) X3(t) and the ideal logic block output (d) z(t).

Similarly, pulse Bz is due to the overlapping between pulses $X2_1$ and $X3_1$, and Cz is due to the overlapping between pulses $X1_2$ and $X2_2$. Finally, pulse Dz is due to the overlapping between 1) pulses $X1_3$ and $X2_2$ 2) pulses $X1_3$, $X2_2$ and $X3_2$, and 3) pulses $X1_3$ and $X3_2$.

Because the p.d.f's associated with each gate have finite domain, the times of occurrence of each rise and fall transition in the physical output $\underline{Z}(t)$ have finite spreads. Let $\underline{u}_{AZ}$, $\underline{u}_{BZ}$, $\underline{u}_{CZ}$, $\underline{u}_{DZ}$ and $\underline{d}_{AZ}$, $\underline{d}_{BZ}$, $\underline{d}_{CZ}$, $\underline{d}_{DZ}$ denote the rise and fall transition times respectively, of the pulses in $\underline{Z}(t)$ corresponding to the pulses Az, Bz, Cz, and Dz in z(t). Assuming the circuit performs as intended, the spreads for the random variables $\underline{u}_{AZ}$ and $\underline{d}_{AZ}$ are discussed next.

The rise of pulse Az occurs when X1(t) = 1, X2(t) = 0 and X3(t) rises. Because X2(t) = 0, gate A2 is inhibited. On the other hand gate A1 is enabled since X1(t) = 1. Therefore, the rise in X3(t) propagates through gates O1, A1, and O2. The minimum rise propagation delay of the AND gates is observed from Fig. 4.6 to be 0.1. Similarly, the minimum rise propagation delay for the OR gates is observed to be 0.2. Hence, the minimum total rise propagation delay in this path is 0.2 + 0.1 + 0.2 = 0.5. In a like manner, the maximum value of the total rise propagation delay is determined to be 0.9 + 0.6 + 0.9 = 2.4. Since the rise in X3(t) responsible for the rise of pulse Az occurs at t = 2, it follows that the spread of $\underline{u}_{AZ}$ is given by

$$2.5 < \underline{u}_{AZ} < 4.4. \qquad (4.61)$$

The fall of pulse Az is due to a fall in X1(t) while X2(t) = 0 and X3(t) = 1. This fall transition propagates through the path containing gates A1 and O2. The minimum delay for fall transitions in the AND gates is seen from Fig. 4.6 to be 0.2 and the minimum delay for fall transitions in the OR gates is 0.3. Thus, the minimum total delay for the fall transition of pulse Az is 0.2 + 0.3 = 0.5. Similarly, the maximum total delay for this fall transition is 0.5 + 1.0 = 1.5. Because the fall in X1(t) responsible for the fall of pulse Az occurs at t = 2.5, it follows that

$$3.0 \le \underline{d}_{AZ} \le 4.0 \ . \tag{4.62}$$

The ranges of transition times in $\underline{Z}(t)$ corresponding to the pulses Bz, Cz, and Dz are obtained in like manner. The results are tabulated in Table 4.5.

TABLE 4.5

Transition	$\underline{u}_{AZ}$	$\underline{d}_{AZ}$	$\underline{u}_{BZ}$	$\underline{d}_{BZ}$	$\underline{u}_{CZ}$	$\underline{d}_{CZ}$	$\underline{u}_{DZ}$	$\underline{d}_{DZ}$
min	2.5	3.0	4.8	5.5	9.5	11.5	13.3	19.5
max	4.4	4.0	6.0	6.5	11.4	12.5	14.5	20.5

These spreads are depicted in Fig. 4.8.

The sample functions in the physical circuit output $\underline{Z}(t)$ differ, one from the other, depending upon the actual gate delays within the circuit. In particular, some pulses expected to appear may not occur and some pulses not expected to appear may occur. To illustrate this

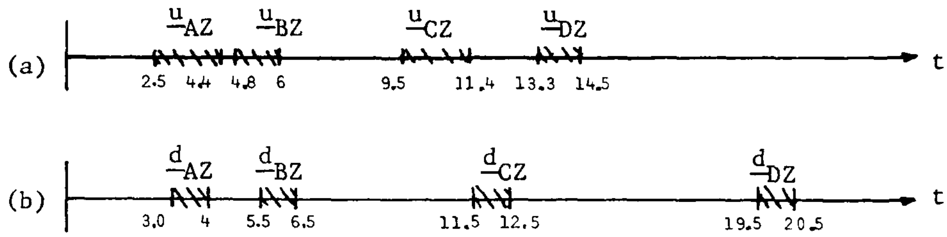


Fig. 4.8 Spreads of transition times in Table 4.8.

point, assume the gate delays are given by the average values $\hat{\tau}_{rA} = 0.35$, $\hat{\tau}_{fA} = 0.35$, $\hat{\tau}_{r0} = 0.55$, $\hat{\tau}_{f0} = 0.65$. Also, assume each gate is modeled as an ideal logic circuit followed by a discriminating delay element. The resulting waveforms $\hat{w}(t)$, $\hat{W}(t)$, $\hat{y}1(t)$, $\hat{Y}1(t)$, $\hat{y}2(t)$, $\hat{Y}2(t)$, $\hat{z}(t)$, $\hat{Z}(t)$ are sketched in Figure 4.9. With reference to Fig. 4.7(d), a pulse $\hat{AZ}$ corresponding to pulse Az is expected to occur in $\hat{Z}(t)$. However, from Fig. 4.8, the rise transition for $\hat{AZ}$ must occur in the interval (2.5, 4.4) while the fall transition must occur in the interval (3,4). Due to the average gate delays, the pulse $\hat{AZ}$ does not occur in $\hat{Z}(t)$, as can be observed by examining the waveforms in Fig. 4.9. The pulses $\hat{BZ}$, $\hat{CZ}$, and $\hat{DZ}$ in $\hat{Z}(t)$ correspond, as expected, to the pulses Bz, Cz, and Dz of z(t). However, the pulses $\hat{EZ}$ and $\hat{FZ}$ deserve special attention. Their rise transitions occur at 8.9 and 20.5 while their fall transitions occur at 9.15 and 21.65. These transitions times are not predicted by Fig. 4.8. Consequently, pulses $\hat{EZ}$ and $\hat{FZ}$ are unintentional output pulses which occur due to the choice of gate delays. It is seen that gate propagation delays can remove intentional pulses and/or add unintentional pulses to the output.

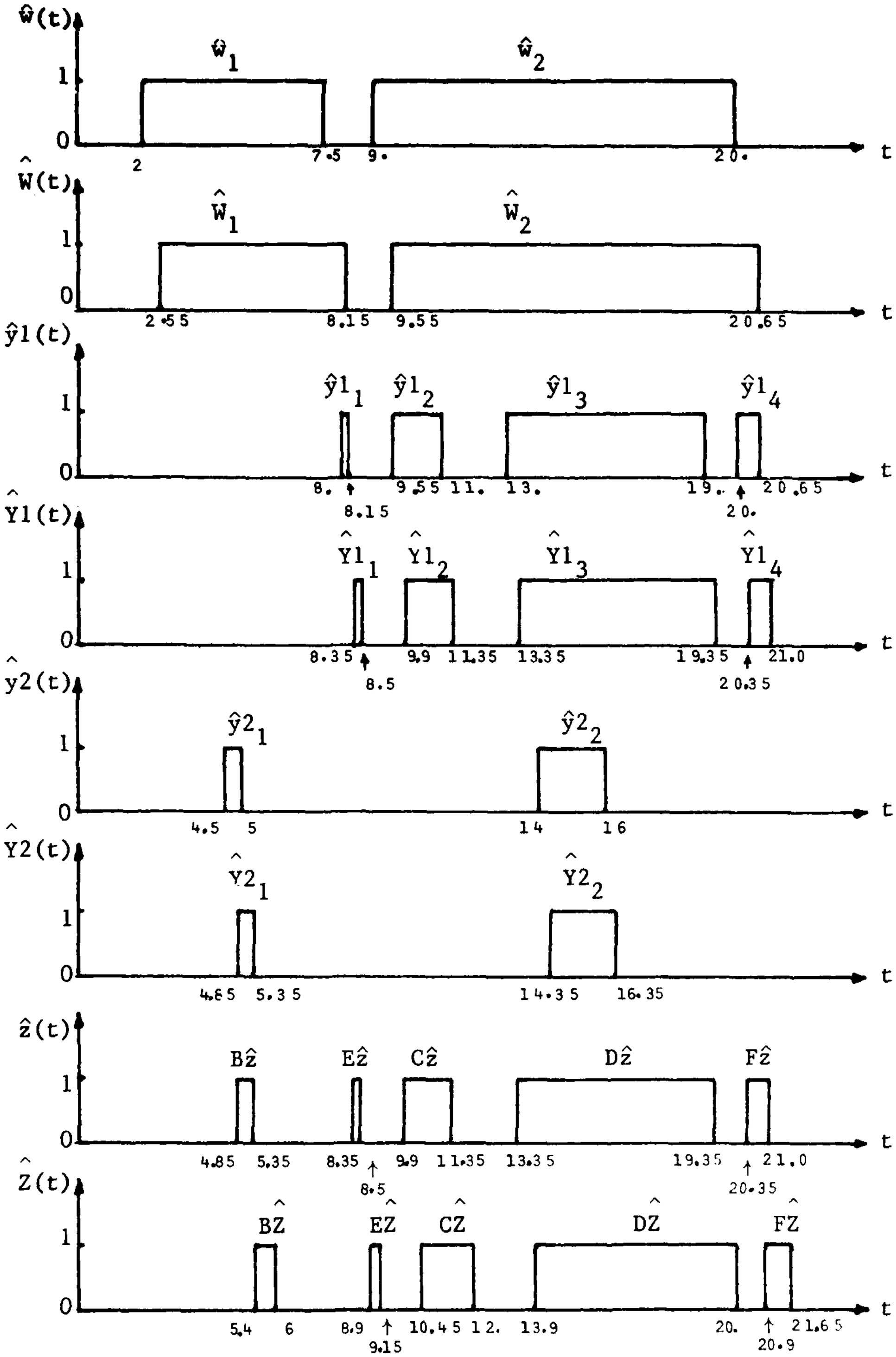


Fig. 4.9. Waveforms in model of circuit in Fig. 4.5 for average gate delays.

Taking into account the spread of the gate delays, it is readily shown that the transition times for pulse EZ and FZ in Z(t) must satisfy the inequalities

$$
\begin{aligned}
8.3 &< \underline{u}_{EZ} < 9.5 \\
8.3 &< \underline{d}_{EZ} < 10.0 \\
20.3 &< \underline{u}_{FZ} < 21.5 \\
20.8 &< \underline{d}_{FZ} < 22.5 \ .
\end{aligned}
\tag{4.63}
$$

The results of the Monte Carlo simulation, $Z_M(t)$, is plotted in Fig. 4.10. Noting the transition times it is possible to distinguish between the intentional and unintentional pulses. Observe that each pulse spans the spread predicted by Table 4.5 and Eqs. (4.63). The intentional pulses are denoted by AZ, BZ, CZ, DZ while the unintentional pulses are denoted by EZ and FZ. The height of each pulse in the Monte Carlo simulation is a measure of its probability of occurrence. It is seen that the intentional pulse AZ rarely occurs. This is also true of the unintentional pulse EZ. Four pulses occur more than 50% of the time. These are the intentional pulses BZ, CZ, and DZ, and the unintentional pulse FZ.

The Monte Carlo simulation serves as the reference waveform for evaluating the four strategies. The evaluation is done by comparing the height, center position, and width of each of the four intentional pulses. The pulse width is defined to be the width at which the estimate of $E[\underline{Z}(t)]$ equals 0.5. No width is given when the height of a pulse is less than 0.5. In general, the height, center position, and width are denoted by H_{IJ}, C_{IJ}, and W_{IJ} where I = A, B, C, or D depending on the pulse and J = M, 1, 2, 3 or 4

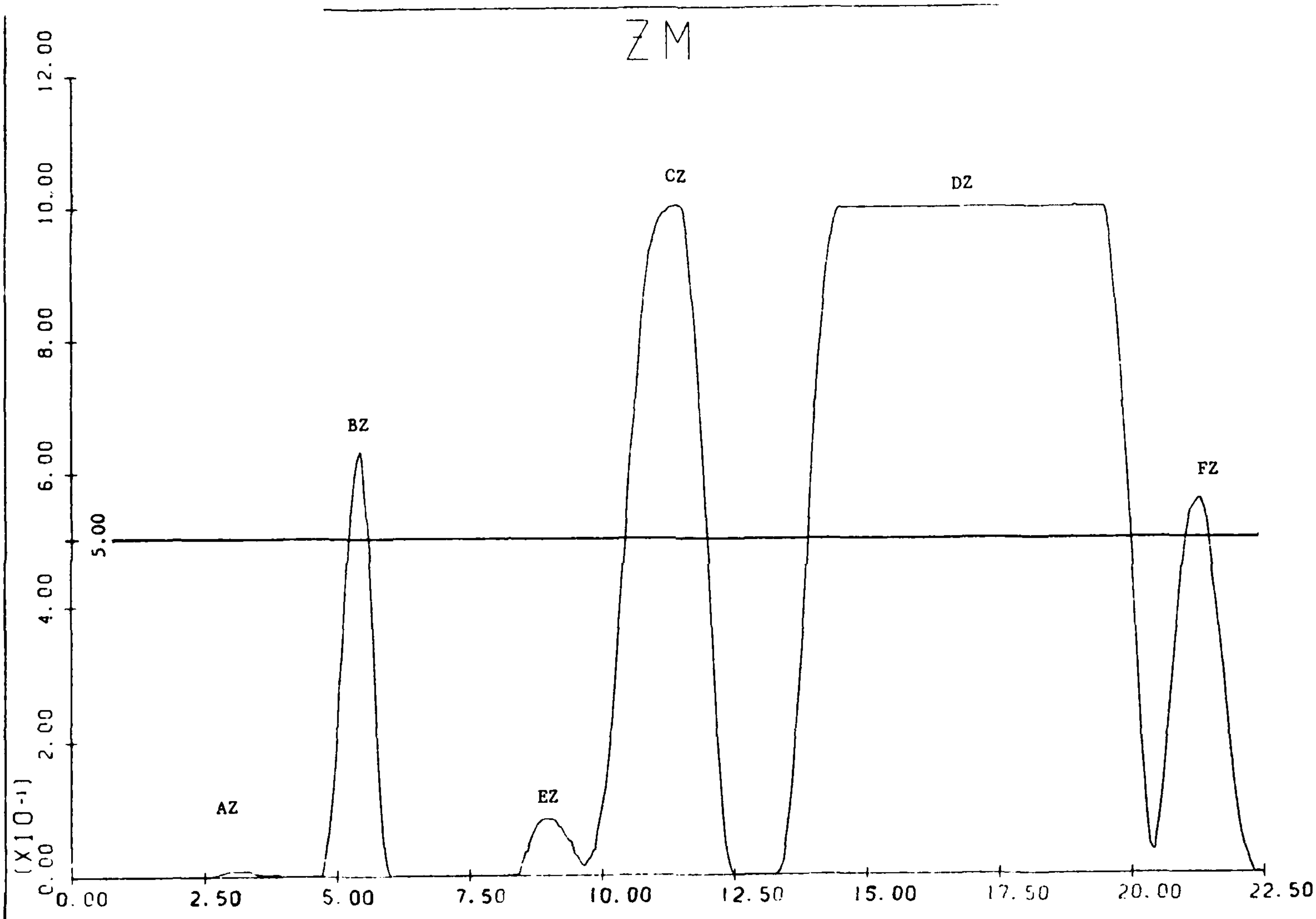


Fig. 4.10. The reference waveform, $Z_M(t)$, obtained by a Monte Carlo simulation.

depending on the procedure used to determine the estimate for $E[\underline{Z}(t)]$. M stands for Monte Carlo simulation while 1, 2, 3, and 4 stand for strategies 1, 2, 3, and 4, respectively. Features of $E[\underline{Z}(t)]$, as estimated from $Z_M(t)$ in Fig. 4.10, are tabulated in Table 4.6.

TABLE 4.6

Pulse IZ	AZ	BZ	EZ	CZ	DZ	FZ
H_{IM}	0.008	0.625	0.086	1.00	1.00	0.55
W_{IM}	-	0.4	-	1.6	6.3	0.4
C_{IM}	3.10	5.35	9.0	11.25	17.0	21.3

The quality of each strategy for characterizing output delays in the simple output delay model is next evaluated with reference to $Z_M(t)$. The output delays for each strategy were assigned and the output expected value of the large logic block was calculated and plotted using the computer. Each strategy is evaluated by comparing the features of each pulse in the calculated expected value with those given by Table 4.6. Since the output of the simple output delay model is a delayed version of z(t), the model cannot predict the unintentional pulses EZ and FZ no matter which strategy is employed. Also, because pulses Az and Bz have the same width of 0.5, as seen in Fig. 4.7(d), pulses AZ and BZ have identical shapes independent of the strategy. These inherent errors in the simple output delay model can get very large.

The computations for each strategy are discussed next where use is made of the results from the examples in Sec. 4.2.

1) <u>The longest path delay</u>. The results of Eqs.(4.3) are used with the particular p.d.f.'s sketched in Fig. 4.6. The resulting p.d.f.'s, $f_{\underline{\tau}rT}(\tau)$ and $f_{\underline{\tau}fT}(\tau)$, and the estimated expected value of the block output $\underline{Z}(t)$, denoted by $E_1[\underline{Z}(t)]$, are plotted in Fig. 4.11. Features of $E_1[\underline{Z}(t)]$, are summarized in Table 4.7.

TABLE 4.7

Pulse IZ	AZ	BZ	CZ	DZ
H_{I1}	0.73	0.73	1.00	1.00
W_{I1}	0.7	0.7	2.2	6.2
C_{I1}	3.9	6.4	11.6	17.6

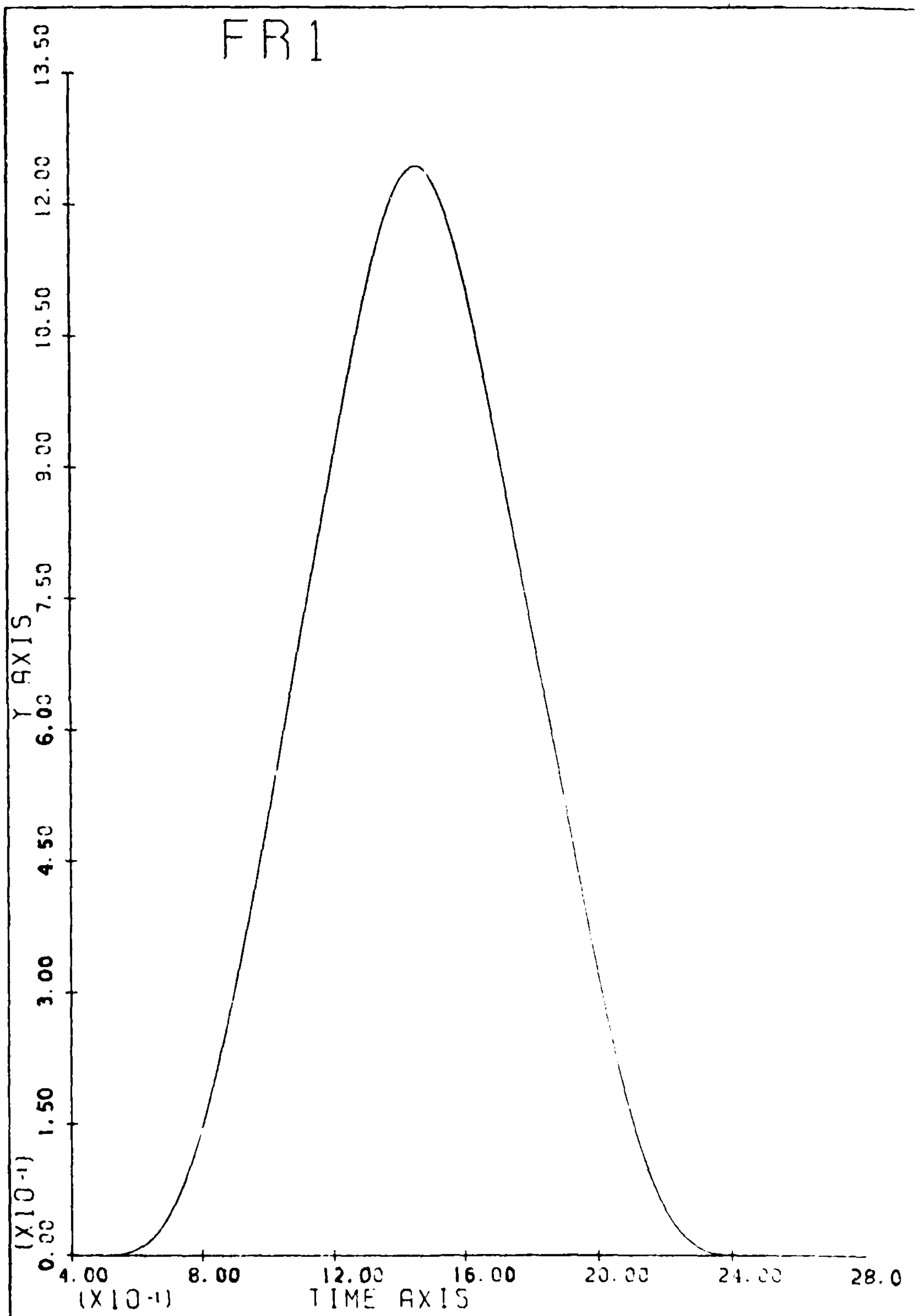
As mentioned earlier, because pulses Az and Bz in the ideal logic block output, z(t), have the same width, pulses AZ and BZ have identical shapes in Fig. 4.11(c) This is an inherent error of the simple output delay model which is also observed in the other three strategies.

The figures of merit used in evaluating the heights and widths of the pulses obtained by the various strategies for the estimates of $E[\underline{Z}(t)]$ are defined to be

$$h_{IJ} = \frac{|H_{IM} - H_{IJ}|}{H_{IM}} \tag{4.64}$$

$$w_{IJ} = \frac{|W_{IM} - W_{IJ}|}{W_{IM}} \tag{4.65}$$

where the subscripts I and J have the same interpretation as before. To obtain a figure of merit for the pulse center position, let C_{Iz} denote



(a)

Fig. 4.11. Longest path delay (a) $f_{\underline{T}rT}(\tau)$, (b) $f_{\underline{T}fT}(\tau)$, (c) $E_1[\underline{Z}(t)]$.

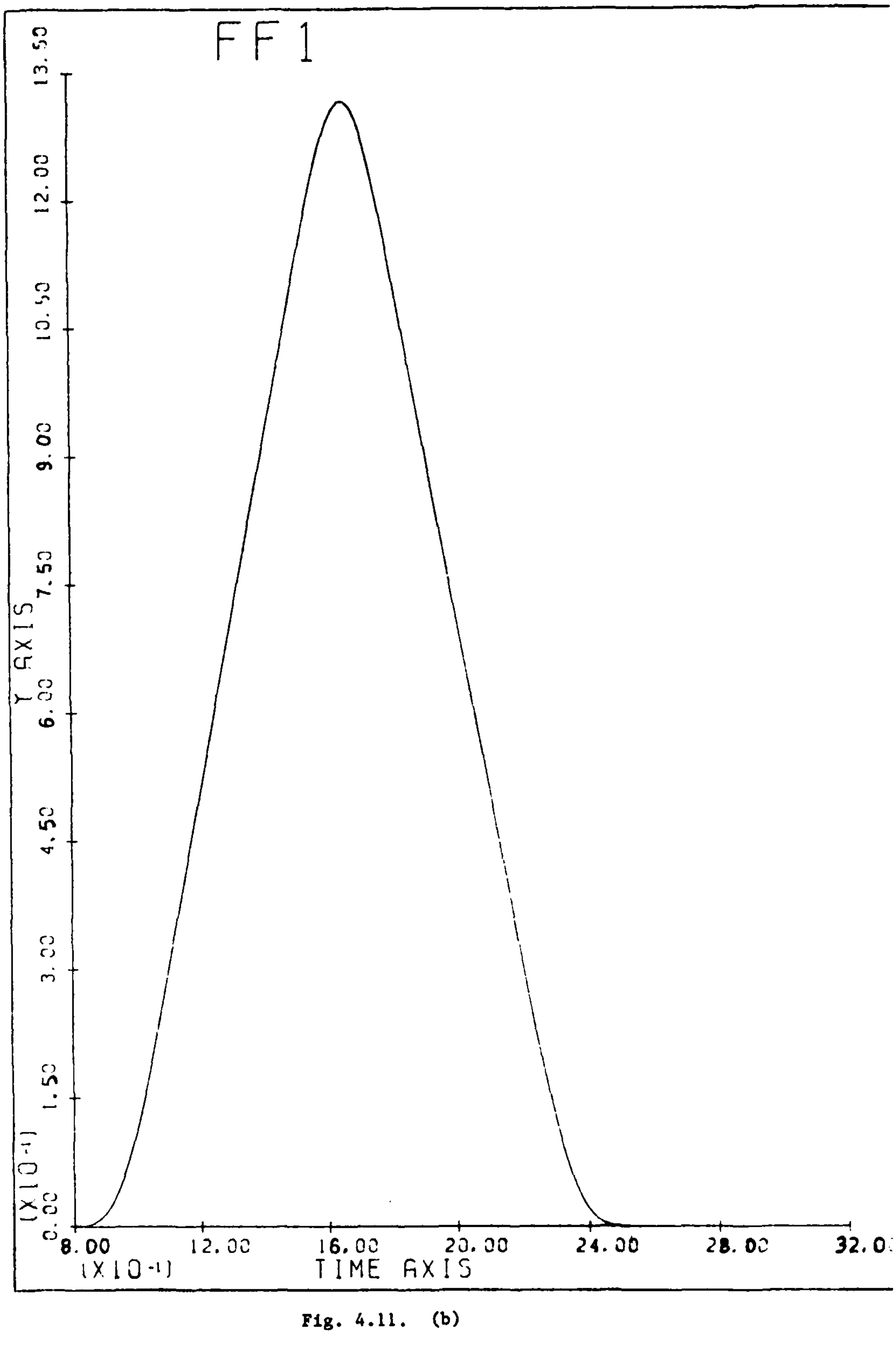

FF1
Y AXIS
(X10-1)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
8.00
12.00
16.00
20.00
24.00
28.00
32.0
(X10-1)
TIME AXIS

Fig. 4.11. (b)

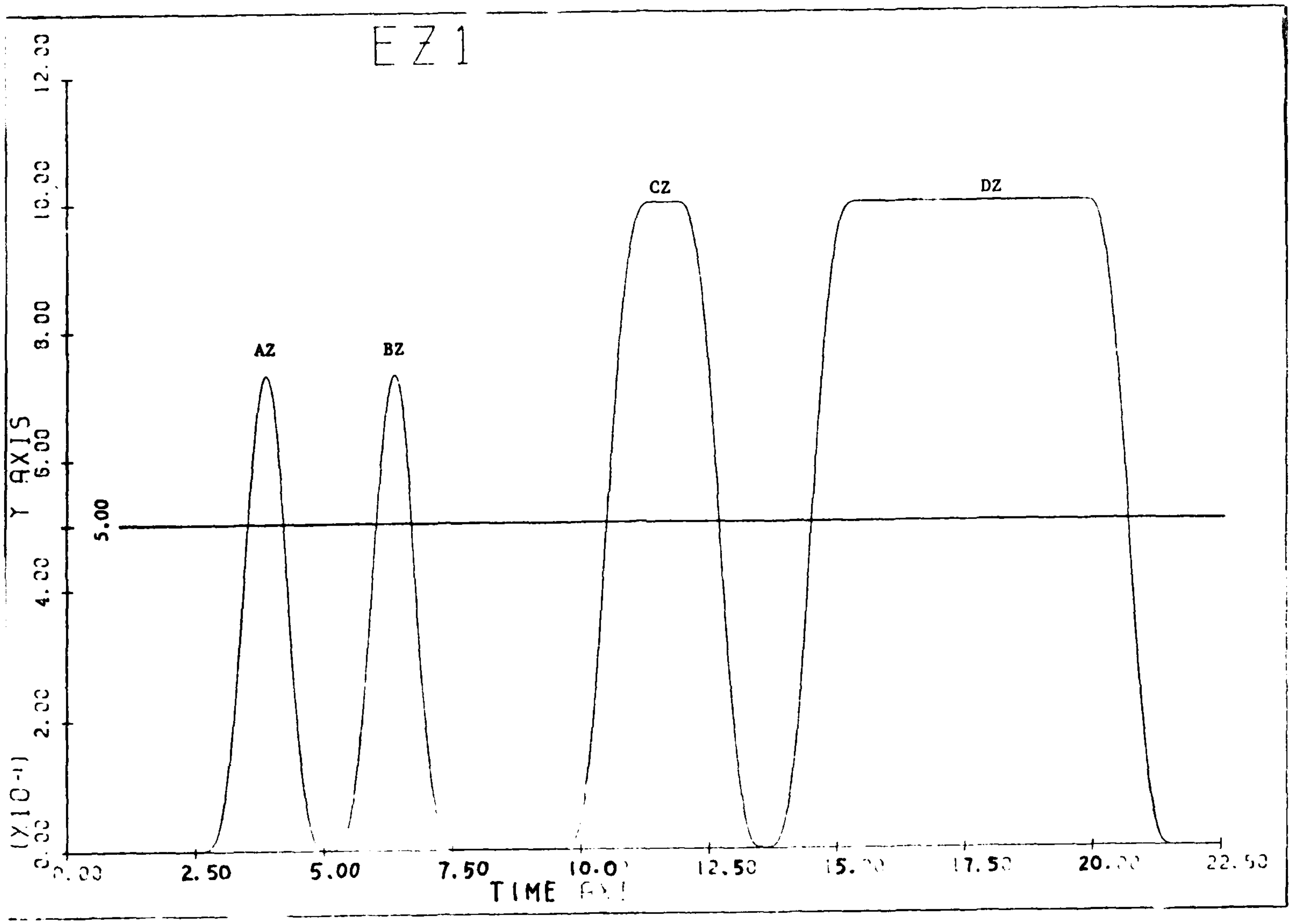

EZ1
AZ
BZ
CZ
DZ
5.00
Y AXIS
(X10-1)
0.00
2.00
4.00
6.00
8.00
10.00
12.00
2.50
5.00
7.50
12.50
20.
22.50
TIME

Fig. 4.11. (c)

the center position of the pulses in the ideal logic output $z(t)$ where, as before, I = A, B, C, or D depending on the pulse. The figure of merit for the pulse center position is defined to be

$$c_{IJ} = \frac{|C_{IM} - C_{IJ}|}{(C_{IM} - C_{Iz})} . \tag{4.67}$$

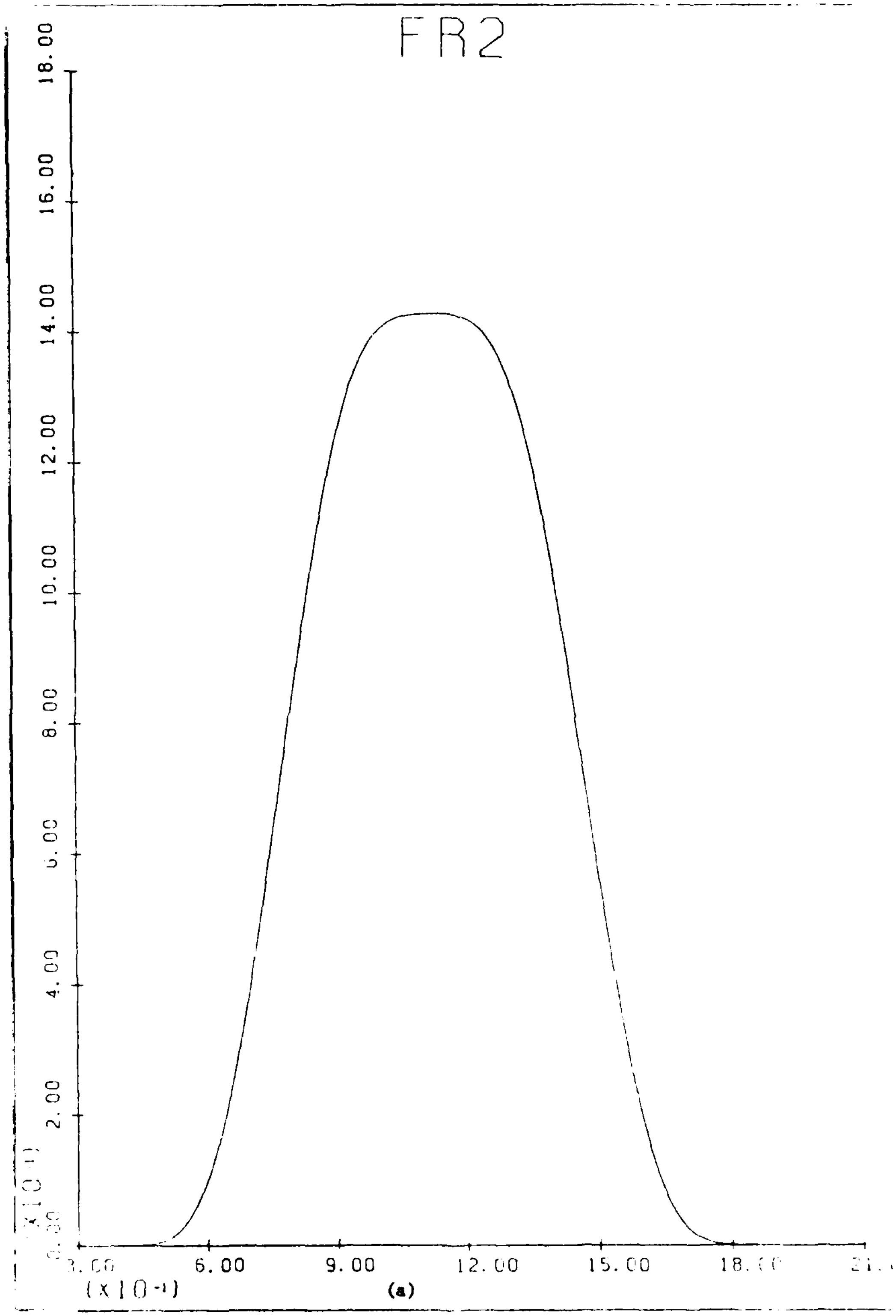
The figures of merit for the longest path delay strategy are tabulated in Table 4.8.

TABLE 4.8

Pulse IZ	AZ	BZ	CZ	DZ
h_{I1}	90.0	0.168	0.	0.
w_{I1}	-	0.75	0.375	0.016
c_{I1}	0.94	0.95	0.28	0.6

Note that the figures of merit are decidedly better for the wide pulses of $z(t)$. Specifically, the figures for the pulses CZ and DZ, are much smaller than those for AZ and BZ. In general, it will be seen for all four strategies that the simple output delay model better simulates the physical circuit as the pulses and gaps in $z(t)$ become longer.

2) <u>The mean path delay</u>. The p.d.f.'s $f_{\underline{\overline{T}}r}(\tau)$ and $f_{\underline{\overline{T}}f}(\tau)$ are determined by carrying out the convolutions in Eqs. (4.13) and (4.14). The results are used to obtain an estimate for the expected value of $\underline{Z}(t)$, denoted by $E_2[\underline{Z}(t)]$. Fig. 4.12 shows the plots of $f_{\underline{\overline{T}}r}(\tau)$, $f_{\underline{\overline{T}}f}(\tau)$, and $E_2[\underline{Z}(t)]$. Features of $E_2[\underline{Z}(t)]$ and the figures of merit are summarized in Table 4.9.



(a)

Fig. 4.12. Mean path delay (a)$f_{\underline{T}r}(\tau)$, (b)$f_{\underline{T}f}(\tau)$, (c)$E_2[\underline{Z}(t)]$.

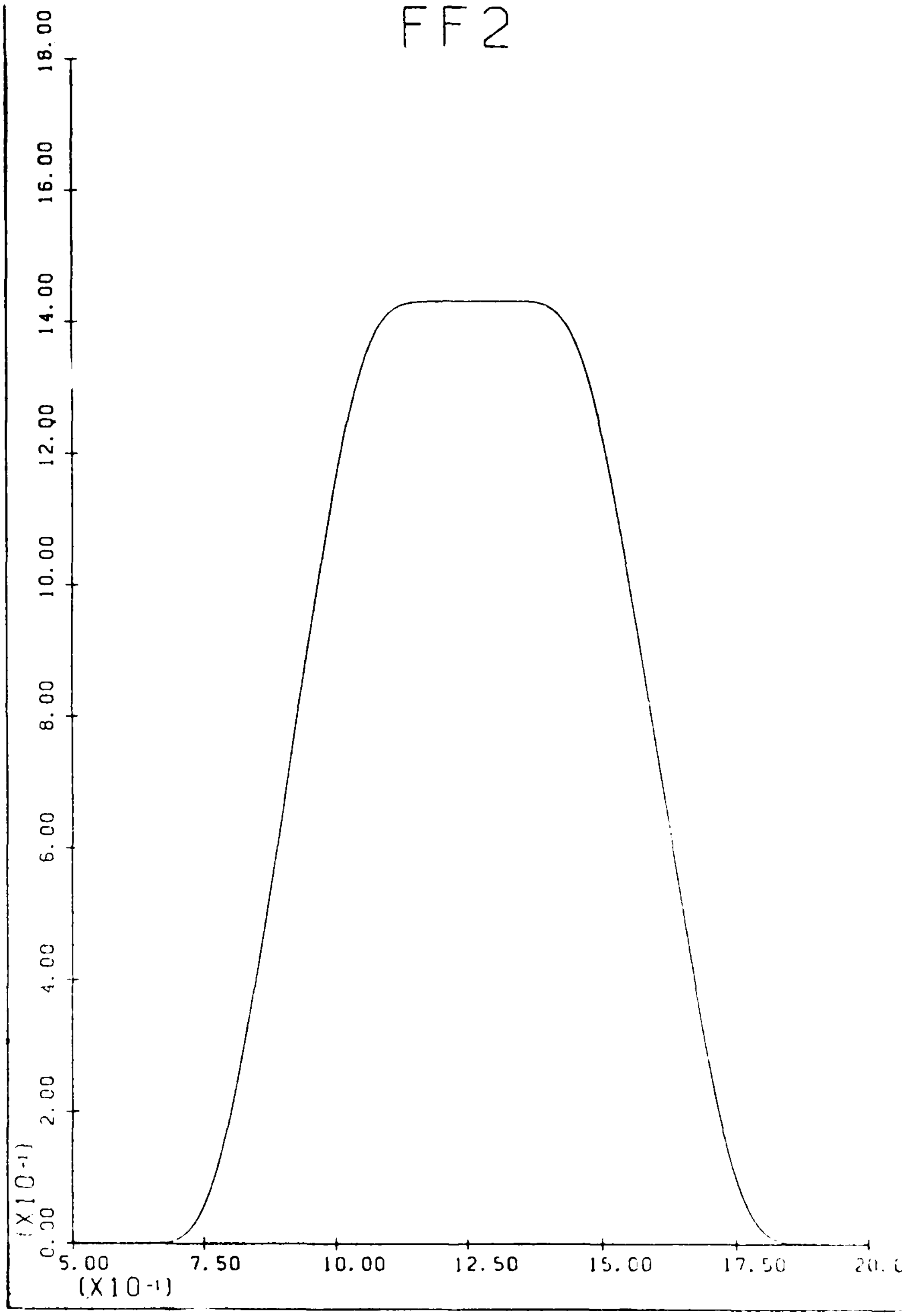
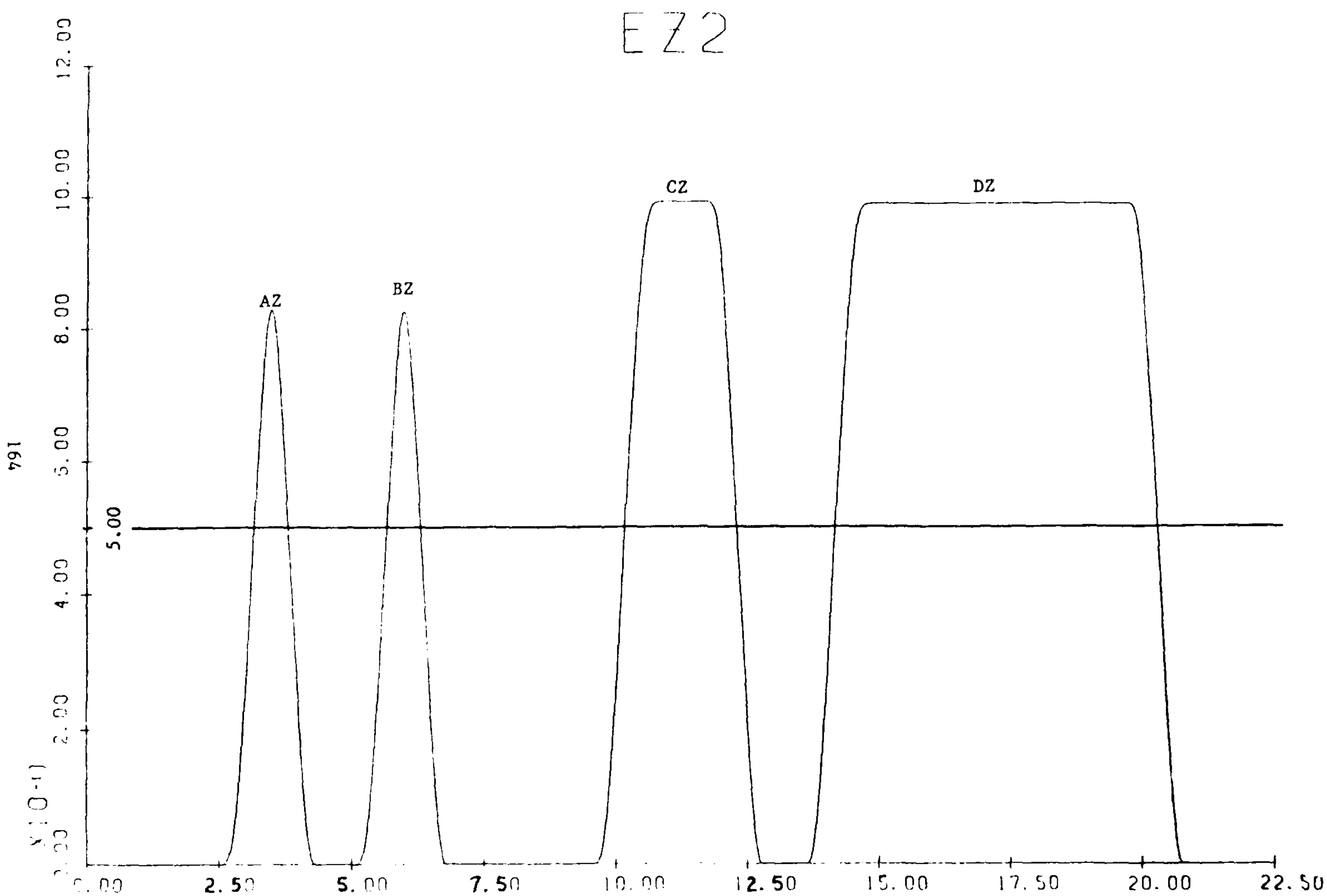

FF2
18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
(X10-1)
5.00
7.50
10.00
12.50
15.00
17.50
(X10-1)

Fig. 4.12. (b)



EZ2
AZ
BZ
CZ
DZ
5.00
12.00
10.00
8.00
6.00
4.00
2.00
X10-1
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50

Fig. 4.12.(c)

TABLE 4.9

Pulse IZ	AZ	BZ	CZ	DZ
H_{I2}	0.825	0.825	1.00	1.00
W_{I2}	0.65	0.65	2.2	6.1
C_{I2}	3.55	6.05	11.3	17.25
h_{I2}	102.0	0.32	0.	0.
w_{I2}	—	0.625	0.375	0.032
c_{I2}	0.53	0.636	0.04	0.25

3) The weighted mean path delay. The input signals X1(t), X2(t) and X3(t), sketched in Fig. 4.7, are used in Eqs. (4.31) and (4.32) to determine the weights a_{rkm} and a_{fkm}. Because the inputs are deterministic signals, the expected values in Eqs. (4.31) and (4.32) are replaced by the signals themselves. For example, $a_{r12} = \langle \dot{X}1^{+}(t)X2'(t)X3(t)\rangle = \frac{1}{24}\int_0^{24} \dot{X}1^{+}(t)X2'(t)X3(t)dt = 0$, since no rise transition in X1(t) occurs when X2(t) = 0 and X3(t) = 1. Also $a_{r13} = \langle \dot{X}1^{+}(t)X2(t)X3'(t)\rangle = \frac{1}{24}\int_0^{24} \dot{X}1^{+}(t)X2(t)X3'(t)dt = \frac{1}{24}\int_0^{24} \delta(t-13)dt = \frac{1}{24}$, because X1(t) has a rise at t = 13 while X2(13) = 1 and X3(13) = 0. Similarly, all other weights are obtained, and the values are tabulated in Table 4.10.

TABLE 4.10

km	12	13	22	23	32	33
a_{rkm}	0	1/24	1/24	1/24	0	1/24
a_{fkm}	2/24	1/24	0	0	1/24	0

Substitution of the results in Tables 4.3 and 4.10 into Eqs. (4.21) and (4.22) yields

$$<\underline{\tau}_r> = \{\frac{1}{24}(\underline{\tau}_{rA1} + \underline{\tau}_{r02}) + \frac{1}{24}(\underline{\tau}_{rA2} + \underline{\tau}_{r02}) + \frac{1}{24}(\underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02})$$

$$+ \frac{1}{24}(\underline{\tau}_{r01} + \underline{\tau}_{rA1} + \underline{\tau}_{r02})\} / (\frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24})$$

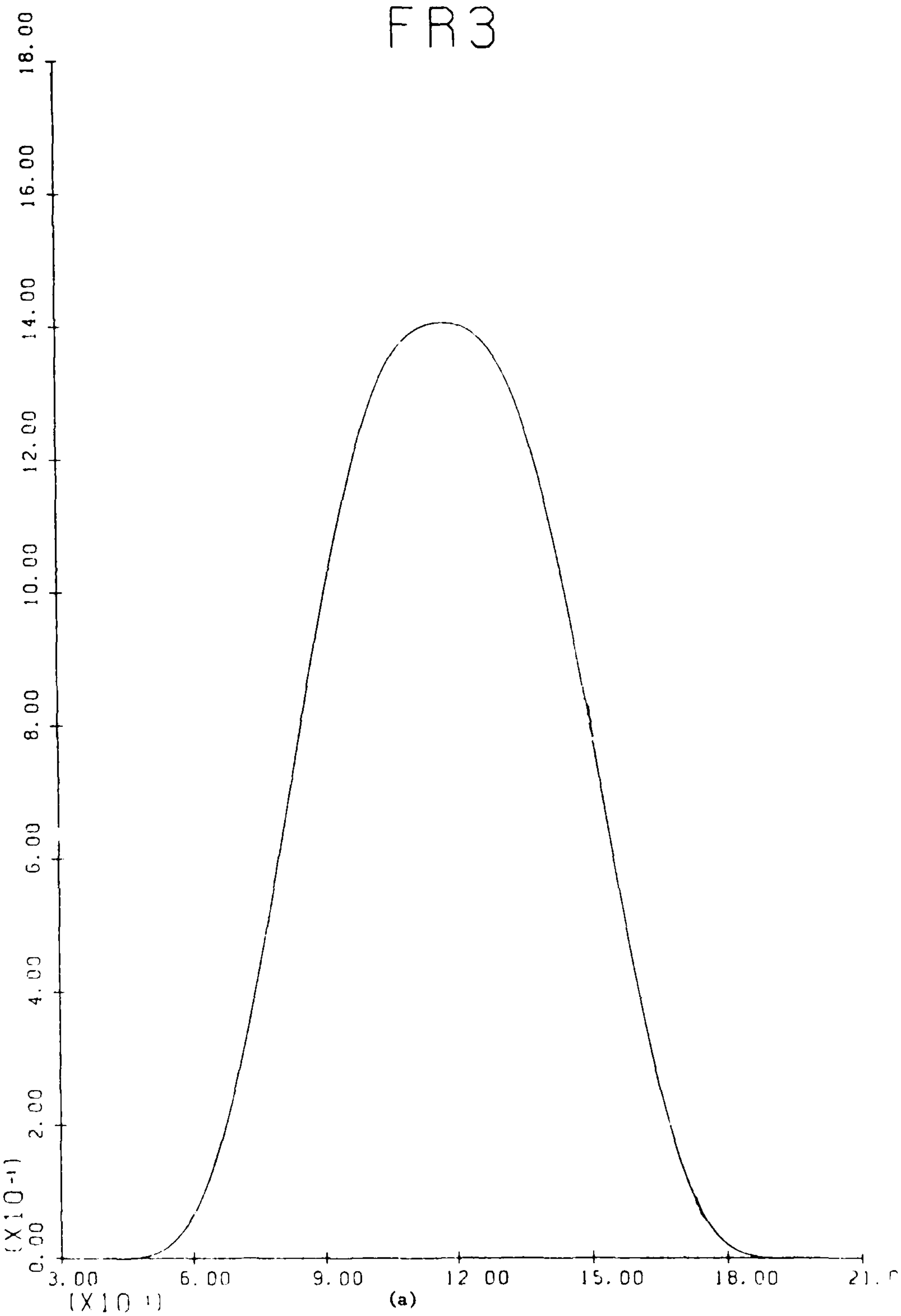
$$= \frac{3}{4}\underline{\tau}_{rA1} + \frac{1}{4}\underline{\tau}_{rA2} + \frac{1}{2}\underline{\tau}_{r01} + \underline{\tau}_{r02}, \tag{4.68}$$

$$<\underline{\tau}_f> = \{\frac{2}{24}(\underline{\tau}_{fA1} + \underline{\tau}_{f02}) + \frac{1}{24}(\underline{\tau}_{fA1} + \underline{\tau}_{f02}) + \frac{1}{24}(\underline{\tau}_{fA2} + \underline{\tau}_{f02})\} /$$

$$(\frac{2}{24} + \frac{1}{24} + \frac{1}{24})$$

$$= \frac{3}{4}\underline{\tau}_{fA1} + \frac{1}{4}\underline{\tau}_{fA2} + \underline{\tau}_{f02} . \tag{4.69}$$

The results of Eqs.(4.68) and (4.69) are used in Eq.(4.6) to derive $f_{<\underline{\tau}r>}(\tau)$ and $f_{<\underline{\tau}f>}(\tau)$, respectively. These p.d.f.'s are used to obtain an estimate for the expected value of the output $\underline{Z}(t)$, denoted by $E_3[\underline{Z}(t)]$. Computer results for $f_{<\underline{\tau}r>}(\tau)$, $f_{<\underline{\tau}f>}(t)$ and $E_3[\underline{Z}(t)]$ are plotted in Fig. 4.13. Features of $E_3[\underline{Z}(t)]$ and the figures of merit are tabulated in Table 4.11.



(a)

Fig. 4.13. Weighted mean path dealy (a)$f_{<\underline{\tau}r>}(\tau)$,(b)$f_{<\underline{\tau}f>}(\tau)$, (c)$E_3[\underline{Z}(t)]$.

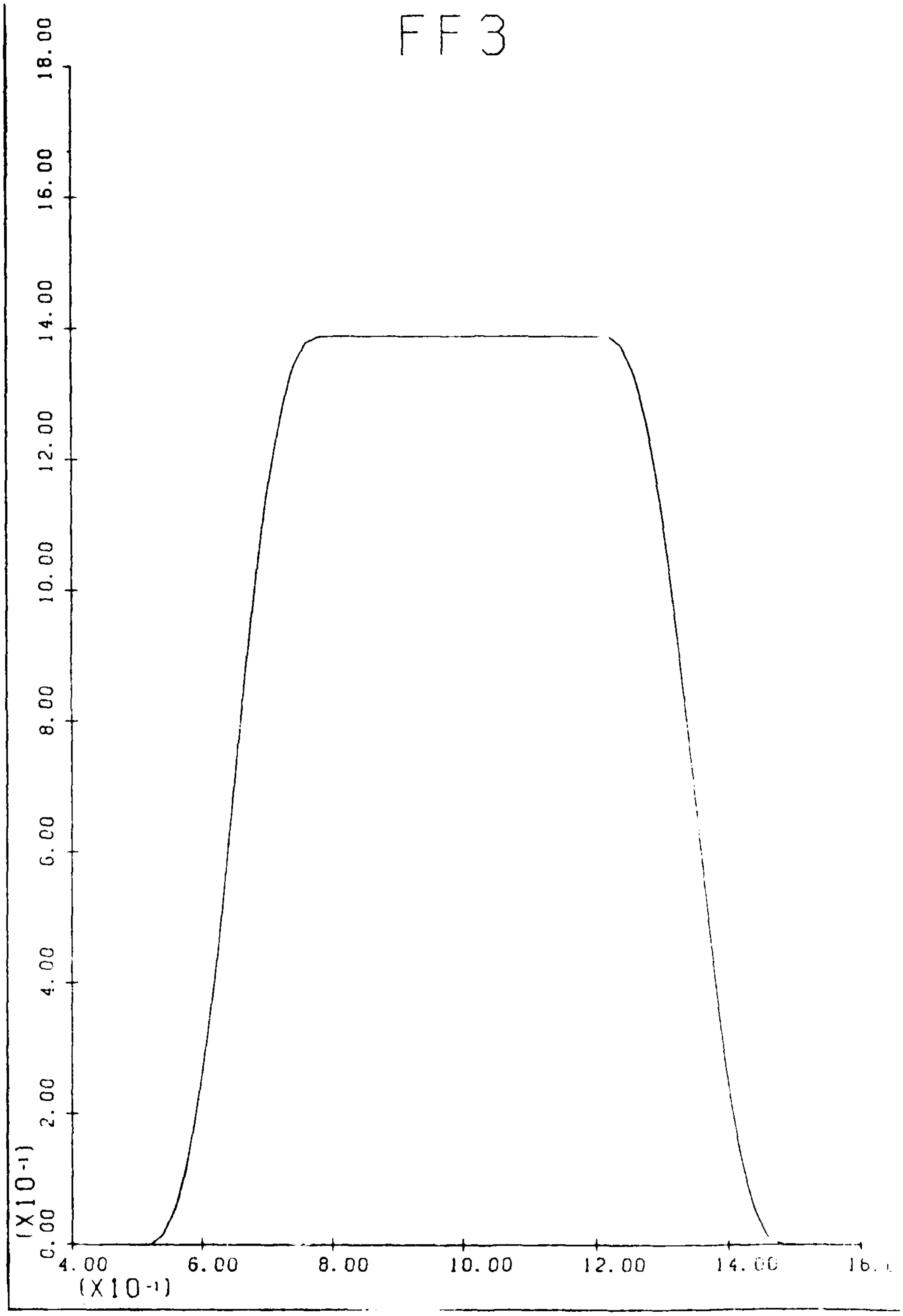

FF3
18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
(X10⁻¹)
4.00
6.00
8.00
10.00
12.00
14.00
(X10⁻¹)

Fig. 4.13. (b)

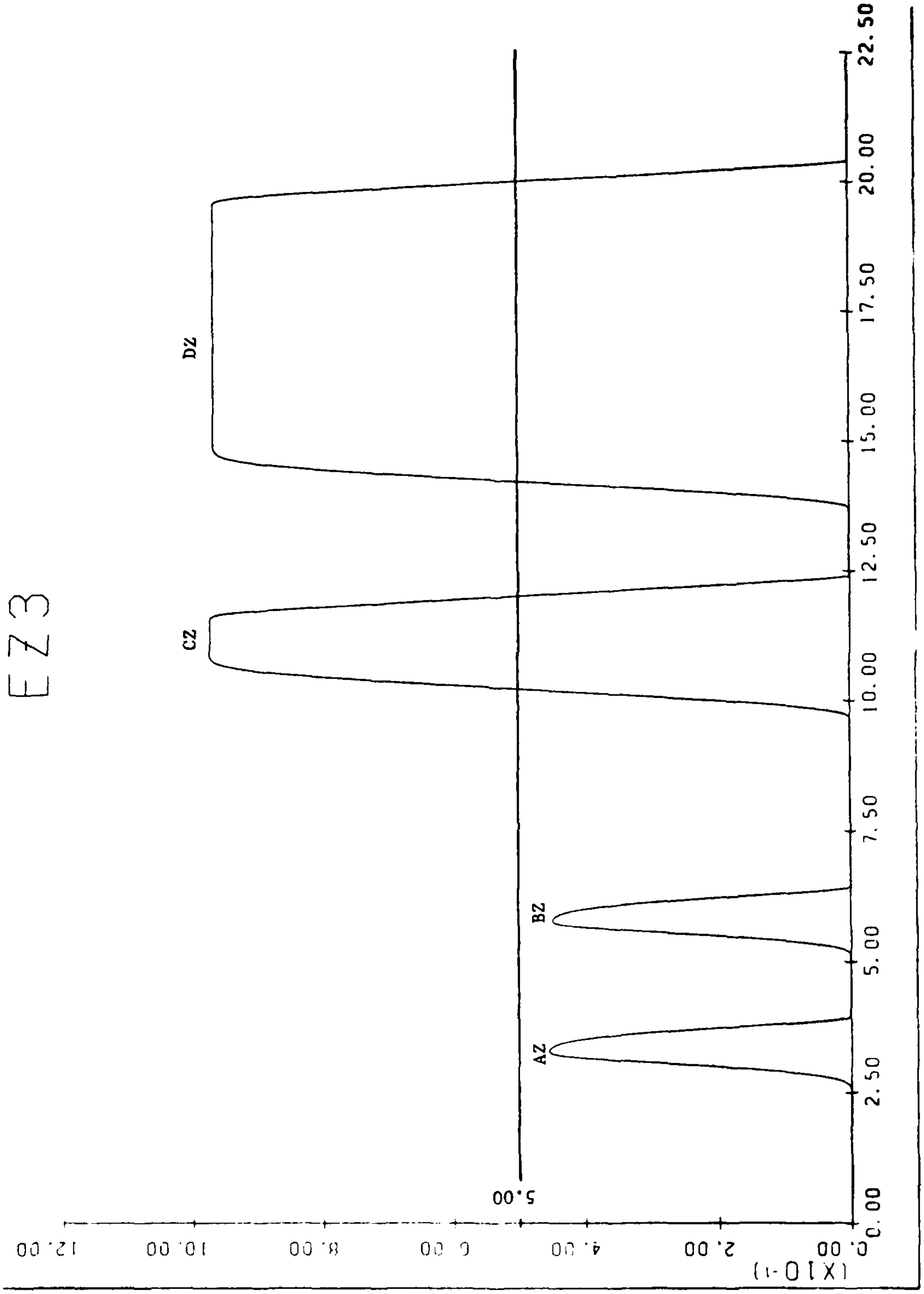

EZ3
AZ
BZ
CZ
DZ
5.00
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50
0.00
2.00
4.00
6.00
8.00
10.00
12.00
(X10-1)

Fig. 4.13.(c)

TABLE 4.11

Pulse IZ	AZ	BZ	CZ	DZ
H_{I3}	0.45	0.45	1.00	1.00
W_{I3}	-	-	1.80	5.90
C_{I3}	3.40	5.85	11.15	17.15
h_{I3}	55.0	0.28	0.0	0.0
w_{I3}	-	-	0.125	0.063
c_{I3}	0.35	0.45	0.08	0.15

4) The Assigned Gaussian delays. Note that the minimum and maximum values of rise and fall propagation delays given by Eqs. (4.41) and (4.42) are the same as the minimum and maximum values for the p.d.f.'s of the delays shown in Fig. 4.6. Therefore, the results of Eqs.(4.47) and (4.48) for $f_{\underline{\tau}rG}(\tau)$ and $f_{\underline{\tau}fG}(\tau)$, respectively, are readily employed as the large logic block output delay p.d.f.'s. These two p.d.f.'s and the estimate of the output expected value, denoted by $E_4[\underline{Z}(t)]$, are plotted in Fig. 4.14. Features of $E_4[\underline{Z}(t)]$ and the figures of merit are tabulated in Table 4.12.

TABLE 4.12

Pulse IZ	AZ	BZ	CZ	DZ
H_{I4}	0.656	0.656	1.00	1.00
W_{I4}	0.6	0.6	2.15	6.1
C_{I4}	3.75	6.25	11.50	17.50
h_{I4}	81	0.05	0.	0.
w_{I4}	-	0.5	0.34	0.03
c_{I4}	0.765	0.82	0.20	0.50

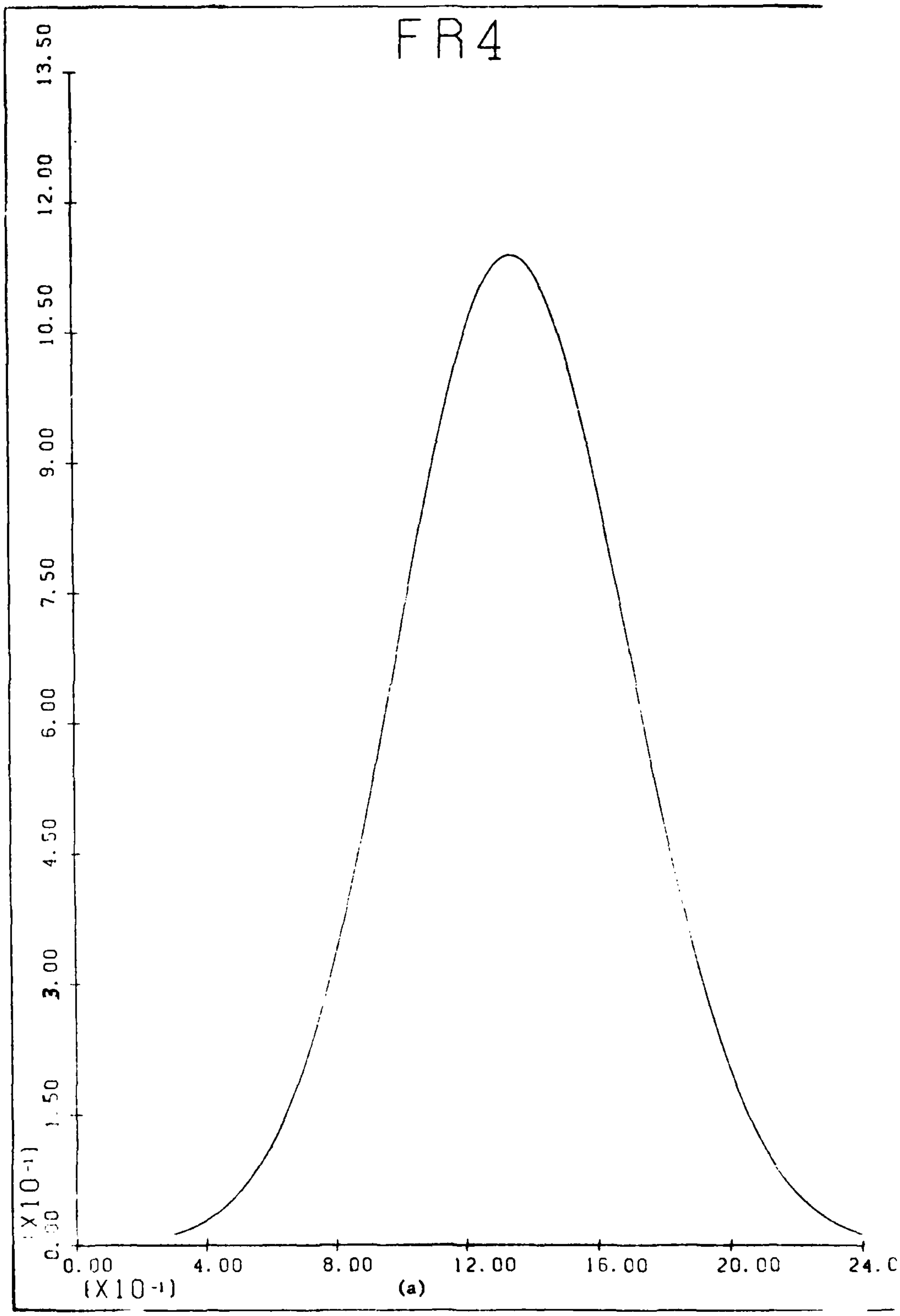


Fig. 4.14. Assigned Gaussian delay (a)$f_{\underline{\tau}rG}(\tau)$, (b)$f_{\underline{\tau}fG}(\tau)$, (c)$E_4[\underline{Z}(t)]$.

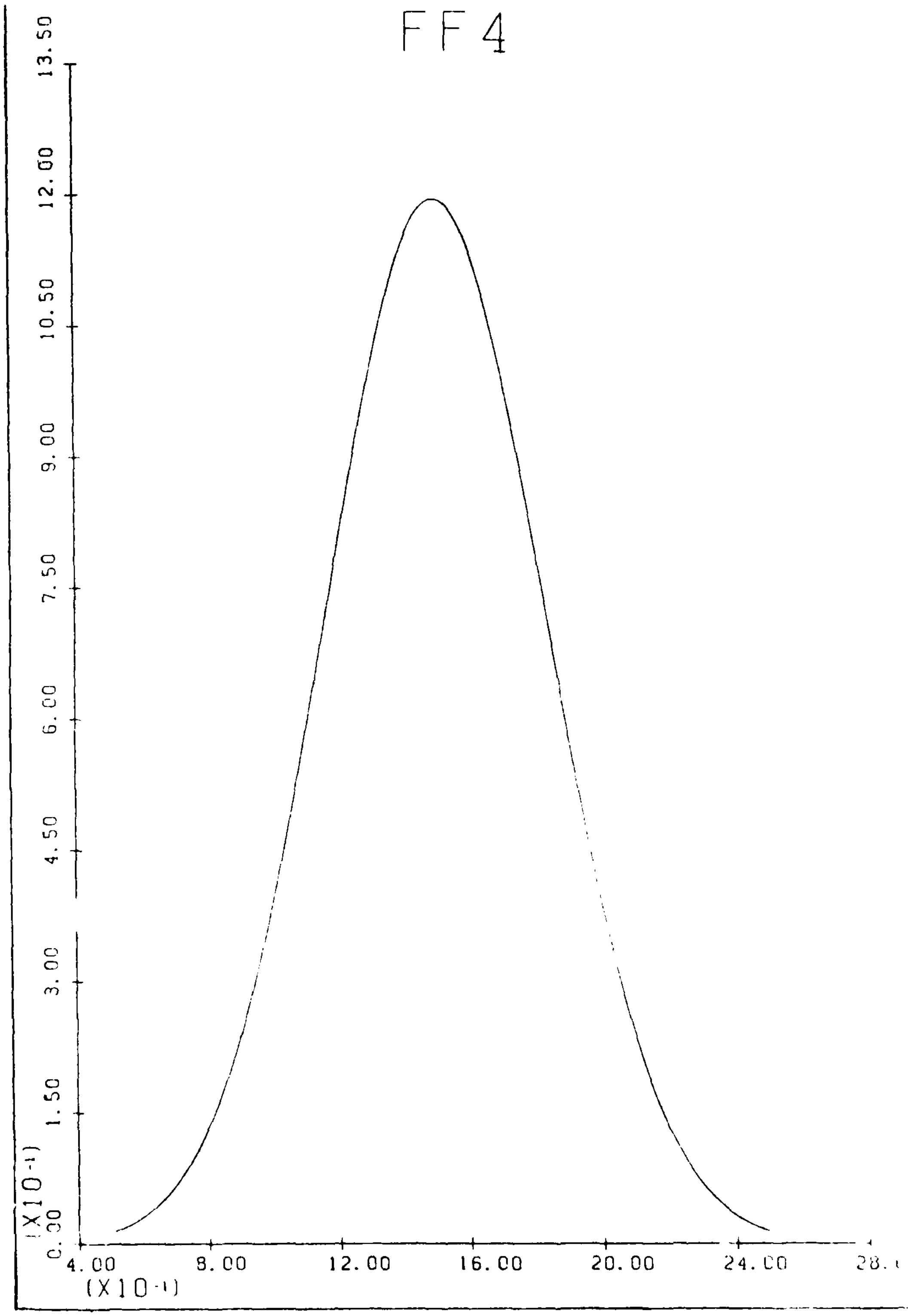

FF4
13.50
12.00
10.50
9.00
7.50
6.00
4.50
3.00
1.50
0.00
(X10⁻¹)
4.00
8.00
12.00
16.00
20.00
24.00
(X10⁻¹)

Fig. 4.14. (b)

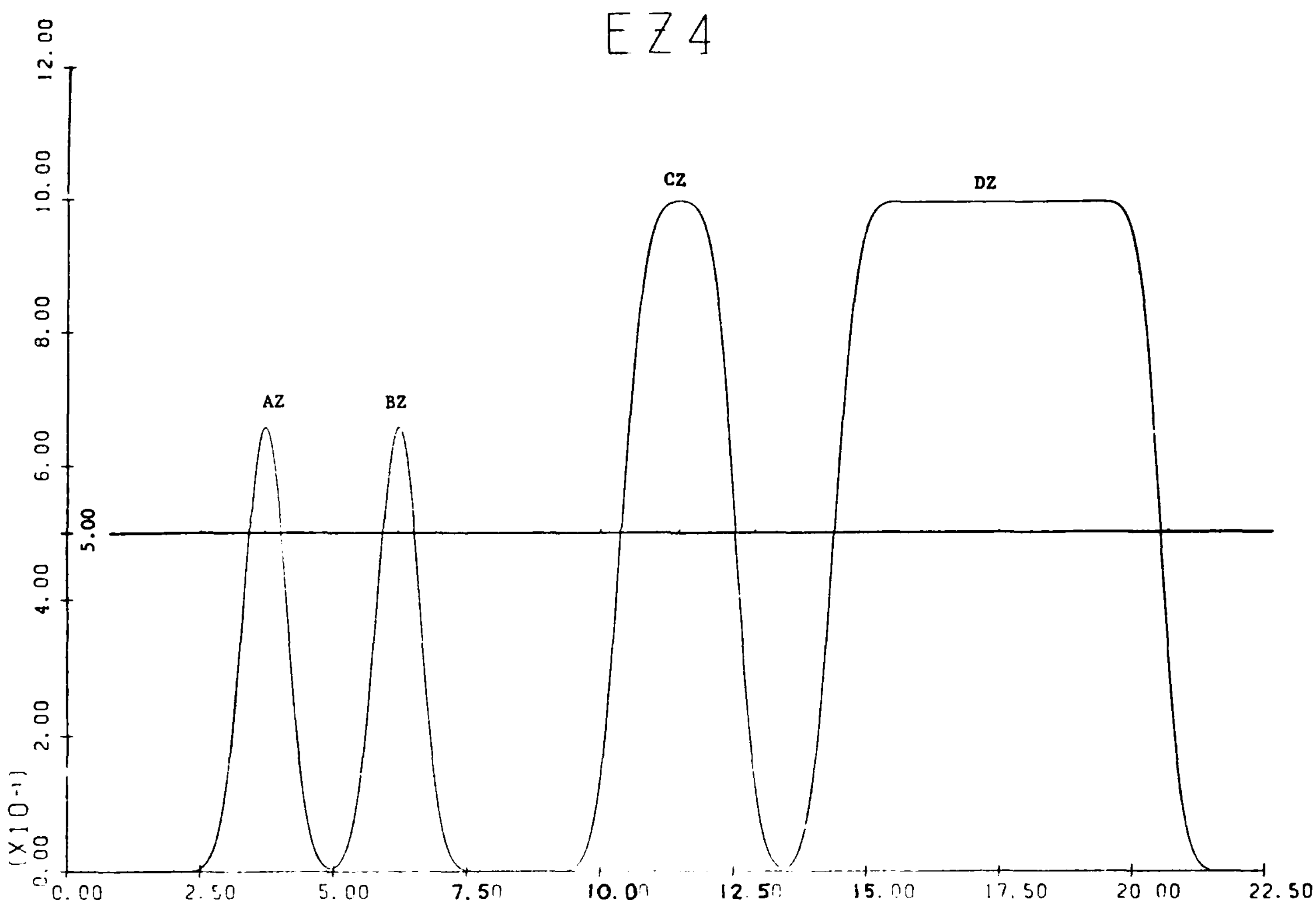

EZ4
AZ
BZ
CZ
DZ
5.00
(X10-1)
0.00
2.00
4.00
6.00
8.00
10.00
12.00
0.00
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
22.50

Fig. 4.14.(c)

Summary. The simple output delay model is seen to have serious short comings. Because no delays are assumed within the ideal logic circuit, phenomena like the nonoccurrence of the intentional pulse AZ and generation of the unintentional pulses EZ and FZ cannot be predicted by the model. Such phenomena occur when durations between transitions in the various input signals are shorter than the spreads of the output delays.

On the other hand, when the durations of pulses and gaps in the ideal logic output, $z(t)$, are sufficiently long, the simple output delay model does a reasonable job of estimating $E[\underline{Z}(t)]$. None of the four strategies yielded decidedly superior performance. The weighted mean delay strategy did slightly better in predicting a smaller probability of occurrence for pulse AZ. However, it also did slightly worse since it predicted a smaller probability of occurrence for pulse BZ. The results of this example tend to suggest that all four strategies perform approximately the same. Since the assigned Gaussian delay strategy is the only one which does not require convolutions, and therefore, is the simplest to implement, it appears to be preferable.

5. SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

5.1. Summary

This dissertation develops a probabilistic analysis for combinational circuits with random delays. A logic block is modeled as an ideal logic circuit, which performs the intended switching function on the block inputs, followed by a delay element, which accounts for propagation delays experienced by the signals. This model is used both for single gates and in the approximation of large logic blocks.

Two kinds of delay elements are discussed. The pure delay element, whose output is a delayed replica of its input, is considered first. The more complicated case of the discriminating delay element, in which input rise and fall transistions experience different delays, is treated next. The output of the discriminating delay element is a distorted version of its input. The amount and complexity of the computations and the required computer storage are significantly larger when logic blocks include discrimating delay elements.

Two types of networks are considered. The simpler type, referred to as tree-like networks have the property that all gate inputs within the network are statistically independent provided the primary inputs to the network are statistically independent. In this case, evaluation of output expected values requires knowledge only of the first order input statistics. In the more complicated type of network, containing reconvergent fanouts, higher order moments of the primary input signals may be required for evaluating output expected values.

In order to simplify computations and reduce the amount of required computer memory an approximate model for a large logic block is suggested.

Different placements of the delay elements in the large logic block model are considered. The simple output delay model is developed and selected as the one most appropriate for EMI applications. Four strategies for assigning p.d.f.'s to the output delay element are proposed. A computer example is used to compare the effectiveness of the four strategies. The reference waveform for the true expected value of the output signal is obtained by means of a Monte Carlo simulation.

5.2 Recommendations for future work.

The probabilistic analysis developed in this work results in extremely complicated expressions, even for simple circuits. The computational efforts required to produce numerical results are considerable. Therefore, if the methods described here are to be applied to practical circuits, it will be necessary to develop an efficient computer code to assist with the analysis. Employing techniques such as used by Debany [10] in obtaining his probability expressions, it may be possible to automatically generate complicated expressions such as appear in appendix B. This would facilitate application of the probabilistic analysis to larger logic circuits. Assuming computational difficulties can be overcome, other problems remain to be studied.

Perhaps the most natural extension of this work is to sequential circuits and digital circuits containing feedback loops. As pointed out by Grundmann [15], signal dependencies caused by either reconvergent fanouts or feedback are somewhat similar in nature. Therefore, techniques developed for combinational circuits with reconvergent fanouts may provide a good starting point for the analysis of sequential circuits and circuits

with feedback. Some preliminary work may be found in references [15], [17], and [24].

An interesting theoretical problem deals with the introduction of blanking effects. Additional knowledge about the signals, such as the statistical properties of the pulse and gap durations, δ_i and υ_j, are needed. In particular, evaluation of p.d.f.'s for $\underline{\delta}_i$ and $\underline{\upsilon}_j$ at a gate output, given the gate switching function and the p.d.f.'s of $\underline{\delta}_i$ and $\underline{\upsilon}_j$ for gate inputs, is required. Introducing the blanking effect into the model eliminates another problem. Recall that rise and fall transitions may be delayed differently. Let a rise be delayed by τ_{r1} and a fall by τ_{f1}, where $\tau_{f1} < \tau_{r1}$. Consider a short pulse of duration δ_i such that $\delta_i < \tau_{r1} - \tau_{f1}$. In this situation, the output of the delay element has the deficiency that the fall transition associated with the pulse occurs before the rise transition. It was pointed out by Grundmann [15] that this effect may cause significant error in evaluation of the output expected value. A dual effect may occur when $\tau_{r2} < \tau_{f2}$ and a short gap of duration υ_j, such that $\upsilon_j < \tau_{f2}-\tau_{r2}$, is introduced to the input of the delay element. In this case the rise of the gap at the delay element output appears before its fall. These two problems are automatically eliminated by including a blanking circuit in the model which rejects any pulse shorter than τ_{r1} (which is certainly larger than $\tau_{r1}-\tau_f$ for any τ_f) and any gap shorter than τ_{f2} (which is clearly larger than $\tau_{f2}-\tau_r$ for any τ_r).

Finally, experimental work is needed to obtain p.d.f's for the delays of the various gates. Measurement of delays under different

environmental and load conditions need to be performed. Analytical expressions for the p.d.f.'s can be derived either from approximations to well known distributions as pointed out in [1, Sec. 9], or by Gaussian approximations, as suggested in [12].

APPENDIX A. EXPECTATION AND CORRELATION OF RANDOM PROCESSES INVOLVING IMPULSES

Throughout this work expected values and correlation functions of 0,1 binary random processes and their time derivatives are evaluated. It was pointed out in Ch.2 that derivatives of sample functions in a 0,1 binary process consist entirely of positive and negative unit area impulses. The mathematical meaning of expectations and correlation functions of processes whose sample functions consist entirely of impulse trains is discussed in this appendix.

The unit area impulse, referred to as the Dirac delta function $\delta(t)$, is characterized by its sampling property [27,p.35]. In other words, if g(t) is any function which is continuous on a neighborhood about the origin, then

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0). \tag{A.1}$$

By a simple change of variable, this property can be readily modified to yield

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} g(\xi+t_o)\delta(\xi)d\xi = g(t_0) \tag{A.2}$$

provided g(t) is continuous in the vicinity of $t=t_0$. There is no ordinary function that has the sampling property. However, the "delta function" can be viewed as a limit of a sequence of functions $\{\delta_n(t)\}$ that, as $n\to\infty$, tend

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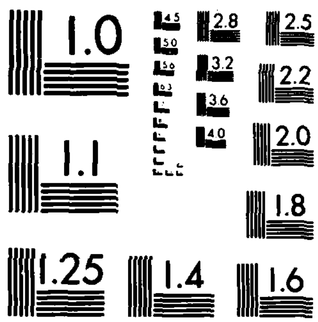

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to zero everywhere except t=0 where they tend to infinity, and the area under each $\delta_n(t)$ is unity [25,Sec.1.5]. As an example of such a sequence, consider

$$\delta_n(t) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2t^2}{2}} \tag{A.3}$$

which is a Guassian function with zero mean and standard deviation equal to $\frac{1}{n}$. This function has unit area and exhibits the desired behavior as $n\to\infty$.

In the theory of generalized functions [26], sequences $\{\gamma_n(t)\}$ are defined such that, for every finite n, the function $\gamma_n(t)$ is infinitely differentiable with respect to t and

$$\lim_{t\to\infty} t^k\gamma_n(t) = 0, \lim_{t\to-\infty} t^k \gamma_n(t) = 0, \quad k > 0. \tag{A.4}$$

Note that the sequence $\{\delta_n(t)\}$ in Eq.(A.3) satisfies these conditions.

A generalized function G(t) is defined by an equivalent sequence $\{\gamma_n(t)\}$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \gamma_n(t)g(t)dt = \int_{-\infty}^{\infty} G(t)g(t)dt. \tag{A.5}$$

In particular, the Dirac delta function is defined by any equivalent sequence having the sampling property. Specifically,

$$\int_{-\infty}^{\infty} \delta(t)g(t)dt \triangleq \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(t)g(t)dt = g(0). \tag{A.6}$$

Two additional important properties of the Dirac delta function are

$$\delta(-t) = \delta(t) \tag{A.7}$$

and

$$\delta(t-t_0)g(t) = \delta(t-t_0)g(t_0) \tag{A.8}$$

for every g(t) which is continous in the vicintiy of $t=t_0$.

The definition of Eq.(A.6) can be generalized to two or more variables [26,.Sec.7.6] by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\theta)\delta(t)g(\theta,t)d\theta dt = \lim_{m\to\infty}\ \lim_{n\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_m(\theta)\delta_n(t)g(\theta,t)d\theta dt = g(0,0) \tag{A.9}$$

for any function of two variables $g(\theta,t)$ which is continuous in the vicinity of the origin (0,0).

The notation $\delta(t)$ is commonly used as a function symbol. However, it has a meaning only as a factor of an integrand, integrated over the variable t. Similarly, the "product" $\delta(\theta)\delta(t)$ is meaningful only as a factor of an integrand under a double integration over the variables θ and t.

Attention is now focused on random variables and stochastic processes. Let $\underline{\tau}$ be a random variable with p.d.f. denoted by $f_{\underline{\tau}}(\tau)$, and let $g(\tau)$ be a function of τ such that the integral $\int_{-\infty}^{\infty} g(\tau)f_{\underline{\tau}}(\tau)d\tau$ exists. The expected value of $g(\tau)$ is given by [18,p.142]

$$E[g(\underline{\tau})] = \int_{-\infty}^{\infty} g(\tau)f_{\underline{\tau}}(\tau)d\tau \tag{A.10}$$

If $\underline{\tau}$ is a random variable of a discrete type, with values taken from the set $\{\tau_i\}$, then

$$E[g(\underline{\tau})] = \sum_i g(\tau_i)P_r\{\ \underline{\tau} = \tau_i\} \tag{A.11}$$

Suppose $\underline{X}(t)$ is a stochastic process such that $\underline{X1}(t) = X(t,\underline{\tau})$. In this case, the source of randomness is the random variable $\underline{\tau}$ and for the specific outcome $\underline{\tau} = \tau$, the sample function $X(t,\tau)$ is deterministic.

Example A.1 Let $\underline{X}(t) = A \sin(\omega t+\underline{\tau})$. Specifying a known value for the random variable $\underline{\tau}$, i.e., setting $\underline{\tau} = \tau$, results in the deterministic sample function $X(t) = A \sin(\omega t+\tau)$. ∇

As in Eq.(A.10), the expected value of $\underline{X}(t) = X(t,\underline{\tau})$ is given by

$$E[\underline{X}(t)] = \int_{-\infty}^{\infty} X(t,\tau) f_{\underline{\tau}}(\tau) d\tau. \tag{A.12}$$

Note that $E[\underline{X}(t)]$ is a deterministic function of time. If $\underline{\tau}$ is a discrete random variable, it follows from Eq.(A.11) that

$$E[\underline{X}(t)] = \sum_i X(t,\tau_i) P_r \{\underline{\tau} = \tau_i\} . \tag{A.13}$$

Let $\underline{X}_d(t)$ be a stochastic process whose sample functions consist of a single impulse occurring at the random time instant $\underline{\tau}$. Specifically,

$$\underline{X}_d(t) = \delta(t-\underline{\tau}) . \tag{A.14}$$

The p.d.f of the random time instant $\underline{\tau}$ is given by $f_{\underline{\tau}}(\tau)$. Direct substitution into Eq. (A.12) results in

$$E[\underline{X}_d(t)] = \int_{-\infty}^{\infty} \delta(t-\tau) f_{\underline{\tau}}(\tau) d\tau . \tag{A.15}$$

Changing the variable of integration yields

$$E[\underline{X}_d(t)] = \int_{-\infty}^{\infty} \delta(\tau) f_{\underline{\tau}}(t-\tau) d\tau. \tag{A.16}$$

Utilizing the sampling property of Eq.(A.1), one readily obtains

$$E[\underline{X}_d(t)] = f_{\underline{\tau}}(t). \tag{A.17}$$

However, the meaning of the expectation of $\underline{X}_d(t)$ is better understood using the sequence approach, as in Eq.(A.6). Application of Eq.(A.6) to Eq.(A.16) results in

$$E[\underline{X}_d(t)] = \int_{-\infty}^{\infty} \delta(\tau) f_{\underline{\tau}}(t-\tau)d\tau = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(\tau) f_{\underline{\tau}}(t-\tau)d\tau. \tag{A.18}$$

Note, for any finite n, that the integrand $\delta_n(\tau)f_{\underline{\tau}}(t-\tau)$ is a finite function of τ for every t and the meaning of the integral is clear. Letting $n\to\infty$ yields the result of Eq.(A.17). If $\underline{\tau}$ is a random variable of a discrete type it follows from Eq.(A.11) that

$$E[\underline{X}_d(t)] = \sum_i \delta(t-\tau_i) P_r\{\underline{\tau} = \tau_i\}. \tag{A.19}$$

Even though $\underline{X}$ (t) consists of a single impulse, the expectation obtained in Eq.(A.19) is an impulse train with impulses occurring at the time instants specified by the set $\{\tau_i\}$ and having areas equal to the corresponding probabilities of occurrence. In this work all expectations evaluated for processes whose sample functions contain impulses are eventually integrated. Consequently, ordinary functions are always obtained from expressions such as that in Eq.(A.19).

The result in Eq.(A.17) is readily generalized to processes having a finite or countable number of impulses in their sample functions. Let the process $\underline{X}_d^N(t)$ have sample functions consisting of N impulses which occur

at the random time instants $\underline{t}_1,\ldots,\underline{t}_N$. The joint p.d.f. of the N random time instants is given by $f_{\underline{t}1\ldots\underline{t}N}(t_1,\ldots,t_N)$. The process $\underline{X}_d^N$ is expressed as

$$\underline{X}_d^N(t) = \sum_{k=1}^{N} \delta(t-\underline{t}_k). \tag{A.20}$$

The expected value of $\underline{X}_d^N(t)$ is given by

$$E[\underline{X}_d^N(t)] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \sum_{k=1}^{N} \delta(t-t_k) f_{\underline{t}1\ldots\underline{t}N}(t_1,\ldots,t_N)dt_1\ldots dt_N. \tag{A.21}$$

Changing the order of summation and integration and using marginal p.d.f.s determined by

$$f_{\underline{t}k}(t_k) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\underline{t}1\ldots\underline{t}N}(t_1,\ldots,t_N)dt_1\ldots dt_{k-1}dt_{k+1}\ldots dt_N, \tag{A.22}$$

result in

$$E[\underline{X}_d^N(t)] = \sum_{k=1}^{N} f_{\underline{t}k}(t). \tag{A.23}$$

Note that Eq.(A.23) is a direct generalization of Eq. (A.17).

The autocorrelation of a stochastic process $\underline{X}(t)$ is defined as

$$R_{\underline{XX}}(t_1,t_2) = E[\underline{X}(t_1)\underline{X}(t_2)] . \tag{A.24}$$

Once again, let $\underline{X}_d(t) = \delta(t-\underline{\tau})$ be the stochastic process whose sample functions consist of a single unit area impulse at a random time instant $\underline{\tau}$.

By definition, the autocorrelation of $\underline{X}_d(t)$ is given by

$$R_{\underline{Xd}\ \underline{Xd}}(t_1,t_2) = E[\delta(t_1-\underline{\tau})\delta(t_2-\underline{\tau})] = \int_{\infty}^{\infty}\delta(t_1-\tau)\delta(t_2-\tau)f_{\underline{\tau}}(\tau)d\tau. \qquad (A.25)$$

Denote $\delta(t_2-\tau)f_{\tau}(\tau)$ by $g(\tau)$. The symmetry property of Eq.(A.7) yields $\delta(t_1-\tau) = \delta(\tau-t_1)$. Application of the sampling property in Eq.(A.1) results in

$$R_{\underline{Xd}\ \underline{Xd}}(t_1,t_2) = \int_{\infty}^{\infty}\delta(\tau-t_1)g(\tau)d\tau = g(t_1)$$

$$= \delta(t_2-t_1)\ f_{\underline{\tau}}(t_1). \qquad (A.26)$$

Note that $R_{\underline{Xd}\ \underline{Xd}}\ (t_1,t_2)$ is an impulse at $t_2=t_1$ with an area equal to $f_{\underline{\tau}}(t_1)$. Note that application of the property in Eq.(A.8) yields

$$\delta(t_2-t_1)f_{\underline{\tau}}(t_1) = \delta(t_2-t_1)f_{\underline{\tau}}(t_2). \qquad (A.27)$$

Hence, as expected, $R_{\underline{Xd}\ \underline{Xd}}\ (t_1,t_2)$ is symmetric with respect to t_1 and t_2. As with expectations, all correlation functions evaluated for processes whose sample functions contain impulses are eventually integrated. Once again, ordinary functions are obtained (see Ex.3.12).

APPENDIX B. EXACT EXPRESSION OF $\dot{E}[\underline{z}^+(t)]$ FOR CIRCUIT USED IN THE COMPUTER SIMULATION

To demonstrate the complexity which may occur in the analytical expression for the output expected value of even a relatively simple circuit, the exact expression of $\dot{E}[\underline{z}^+(t)]$ for the circuit used in the computer simulation of Sec. 4.3 is given below. This is expressed in terms of the circuit inputs, $\underline{X1}(t)$, $\underline{X2}(t)$, and $\underline{X3}(t)$, their counting signals and their derivatives. Statistical independence is assumed among all three inputs.

$$\dot{E}[\underline{z}^+(t)] = \dot{E}[\underline{X1}^+(t-\tau_{rA1})]E[\underline{X2}(0)]E[\underline{X3}(0)] + \dot{E}[\underline{X1}^+(t-\tau_{rA1})]$$

$$\int_0^{t-\tau_{r01}-\tau_{rA1}} R_{\underline{X2X2}'}(0,\theta)R_{\underline{X3\dot{X}3}^+}(0,\theta)d\theta + \dot{E}[\underline{X1}^+(t-\tau_{rA1})]$$

$$\int_0^{t-\tau_{r01}-\tau_{rA1}} R_{\underline{X2\dot{X}2}^+}(0,\theta)R_{\underline{X3X3}'}(0,\theta)d\theta - \dot{E}[\underline{X1}^+(t-\tau_{rA1})]$$

$$\int_0^{t-\tau_{f01}-\tau_{rA1}} R_{\underline{X2X2}'}(0,\theta)R_{\underline{X3\dot{X}3}^-}(0,\theta)d\theta - E[\underline{X1}^+(t-\tau_{rA1})$$

$$\int_0^{t-\tau_{f01}-\tau_{rA1}} R_{\underline{X2\dot{X}2}^-}(0,\theta)R_{\underline{X3X3}'}(0,\theta)d\theta + E[\underline{X1}(t-\tau_{rA1})]$$

$$R_{\underline{X2X2}'}(0,t-\tau_{r01}-\tau_{rA1})R_{\underline{X3\dot{X}3}^+}(0,t-\tau_{r01}-\tau_{rA1})$$

$$+ E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^+}(0,t-\tau_{r01}-\tau_{rA1})R_{\underline{X3X3}'}(0,t-\tau_{r01}-\tau_{rA1})$$

$$+\int_0^{t-\tau_{rA2}}\int_0^{t-\tau_{r01}-\tau_{rA1}}\int_0^{t-\tau_{f01}-\tau_{rA1}}\{\dot{E}[\underline{X1}^+(t-\tau_{rA1})R_{\underline{X2X2}}(0,\theta)\dot{E}[\underline{X3}^+(\theta)]$$

$$+E[\underline{X1}^+(t-\tau_{rA1})]\ R_{\underline{X2\dot{X}2}^+}(0,\theta)E[\underline{X3}(\theta)] + \dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,0)R_{\underline{X3\dot{X}3}^+}(0,\theta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^+}(0,\theta)R_{\underline{X3X3}}(0,\theta) + \dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,\eta)R_{\underline{\dot{X}3}^+\underline{\dot{X}3}^+}(\theta,\eta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^+}(\eta,\theta)R_{\underline{X3\dot{X}3}^+}(\theta,\eta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^+}(\theta,\eta)\ R_{\underline{X3}'\underline{\dot{X}3}^+}(\eta,\theta) + \dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{\dot{X}2}^+\underline{\dot{X}2}^+}(\theta,\eta)R_{\underline{X3X3}'}(\theta,\eta)$$

$$-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,\zeta)R_{\underline{\dot{X}3}^+\underline{\dot{X}3}^-}(\theta,\zeta) - \dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^+}(\zeta,\theta)R_{\underline{X3\dot{X}3}^-}(\theta,\zeta)$$

$$-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{\dot{X}2}\ \underline{\dot{X}2}^-}(\theta,\zeta)R_{\underline{X3}'\underline{\dot{X}3}^-}(\zeta,\theta)-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{\dot{X}2}^+\underline{\dot{X}2}^-}(\theta,\zeta)R_{\underline{X3X3}'}(\theta,\zeta)$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,t-\tau_{r01}-\tau_{rA1})R_{\underline{\dot{X}3}^+\underline{\dot{X}3}^+}(\theta,t-\tau_{r01}-\tau_{rA1})$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^+}(t-\tau_{r01}-\tau_{rA1},\theta)R_{\underline{X3\dot{X}3}^+}(\theta,t-\tau_{r01}-\tau_{rA1})$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^+}(\theta,t-\tau_{r01}-\tau_{rA1})R_{\underline{X3}'\underline{\dot{X}3}^+}(t-\tau_{r01}-\tau_{rA1},\theta)$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{\dot{X}2}^+\underline{\dot{X}2}^+}(\theta,t-\tau_{r01}-\tau_{rA1})R_{\underline{X3X3}'}(\theta,t-\tau_{r01}-\tau_{rA1})\}d\zeta d\eta d\theta$$

$$- \int_0^{t-\tau_{fA2}} \int_0^{t-\tau_{r01}-\tau_{rA1}} \int_0^{t-\tau_{f01}-\tau_{rA1}} \{\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}}(0,\theta)\dot{E}[\underline{X3}^-(\theta)]$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^-}(0,\theta)E[\underline{X3}(\theta)]+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,0)R_{\underline{X3\dot{X}3}^-}(0,\theta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})R_{\underline{X2}'\underline{\dot{X}2}^-}(0,\theta)R_{\underline{X3X3}}(0,\theta) + \dot{E}[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,\eta)R_{\underline{\dot{X}3}^+\underline{\dot{X}3}^-}(\eta,\theta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^-}(\eta,\theta)R_{\underline{X3\dot{X}3}^+}(\theta,\eta)+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^+}(\theta,\eta)R_{\underline{X3}'\underline{\dot{X}3}^-}(\eta,\theta)$$

$$+\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{\dot{X}2}^+\underline{\dot{X}2}^-}(\eta,\theta)R_{\underline{X3X3}'}(\theta,\eta)-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,\zeta)R_{\underline{\dot{X}3}^-\underline{\dot{X}3}^-}(\theta,\zeta)$$

$$-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^-}(\zeta,\theta)R_{\underline{X3\dot{X}3}^-}(\theta,\zeta)-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^-}(\theta,\zeta)R_{\underline{X3}'\underline{\dot{X}3}^-}(\zeta,\theta)$$

$$-\dot{E}[\underline{X1}^+(t-\tau_{rA1})]R_{\underline{\dot{X}2}^-\underline{\dot{X}2}^-}(\theta,\zeta)R_{\underline{X3X3}'}(\theta,\zeta)$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2X2}'}(\theta,t-\tau_{r01}-\tau_{rA1})R_{\underline{\dot{X}3}^+\underline{\dot{X}3}^-}(t-\tau_{r01}-\tau_{rA1},\theta)$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2}'\underline{\dot{X}2}^-}(t-\tau_{r01}-\tau_{rA1},\theta)R_{\underline{X3\dot{X}3}^+}(\theta,t-\tau_{r01}-\tau_{rA1})$$

$$+E[\underline{X1}(t-\tau_{rA1})]R_{\underline{X2\dot{X}2}^+}(\theta,t-\tau_{r01}-\tau_{rA1})R_{\underline{X3}'\underline{\dot{X}3}^-}(t-\tau_{r01}-\tau_{rA1},\theta)$$

$$+ E[\underline{X1}(t-\underline{\tau}_{rA1})R_{\underline{\dot{X2}}^+\underline{\dot{X2}}^+}(t-\underline{\tau}_{r01}-\underline{\tau}_{rA1},\theta)R_{\underline{X3X3}'}(\theta,t-\underline{\tau}_{r01}-\underline{\tau}_{rA1})\} \, d\zeta d\eta d\theta$$

$$+ E[\underline{X1}(0)]R_{\underline{X2X2}}(0,t-\underline{\tau}_{rA2})\dot{E}[\underline{X3}^+(t-\underline{\tau}_{rA2})]+E[\underline{X1}(0)]R_{\underline{X2\dot{X}2}^+}(0,t-\underline{\tau}_{rA2})E[\underline{X3}(t-\underline{\tau}_{rA2})]$$

$$+ E[\underline{X1}(0)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA},0)R_{\underline{X3\dot{X}3}^+}(0,t,\underline{\tau}_{rA2})+E[X1(0)]R_{\underline{X2}'\underline{\dot{X}2}^+}(0,t-\underline{\tau}_{rA2})R_{\underline{X3X3}}(0,t-\underline{\tau}_{rA2})$$

$$+ \int_0^{t-\underline{\tau}_{rA1}} \int_0^{\theta-\underline{\tau}_{r01}} \int_0^{\theta-\underline{\tau}_{f01}} \{\dot{E}[X1^+(\theta)]R_{\underline{X2X2}}(0,t-\underline{\tau}_{rA2})\dot{E}[\underline{X3}^+(t-\underline{\tau}_{rA2})$$

$$+ \dot{E}[\underline{X1}^+(\theta)]R_{\underline{X2\dot{X}2}^+}(0,t-\underline{\tau}_{rA2})E[\underline{X3}(t-\underline{\tau}_{rA2})]$$

$$+\dot{E}[\underline{X1}^+(\theta)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},0)R_{\underline{X3\dot{X}3}^+}(0,t-\underline{\tau}_{rA2})+E[\underline{X1}^+(\theta)]R_{\underline{X2}'\underline{\dot{X}2}^+}(0,t-\underline{\tau}_{rA2})R_{\underline{X3X3}}(0,t-\underline{\tau}_{rA2})$$

$$+ \dot{E}[\underline{X1}^+(\theta)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},\eta)R_{\underline{\dot{X3}}^+\underline{\dot{X3}}^+}(t-\underline{\tau}_{rA2},\eta)$$

$$+ \dot{E}[\underline{X1}^+(\theta)]R_{\underline{X2}'\underline{\dot{X}2}^+}(\eta,t-\underline{\tau}_{rA2})R_{\underline{X3\dot{X}3}^+}(t-\underline{\tau}_{rA2},\eta)$$

$$+ \dot{E}[\underline{X1}^+(\theta)R_{\underline{X2\dot{X}2}^+}(t-\underline{\tau}_{rA2},\eta)R_{\underline{X3}'\underline{\dot{X}3}^+}(\eta,t-\underline{\tau}_{rA2})$$

$$+ \dot{E}[\underline{X1}^+(\theta)]R_{\underline{\dot{X2}}^+\underline{\dot{X2}}^+}(t-\underline{\tau}_{rA2},\eta)R_{\underline{X3X3}'}(t-\underline{\tau}_{rA2}\eta)$$

$$- \dot{E}[\underline{X1}^+(\theta)R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},\zeta)R_{\underline{\dot{X3}}^+\underline{\dot{X3}}^-}(t-\underline{\tau}_{rA2},\eta)$$

$$- \dot{E}[\underline{X1}^+(\theta)]R_{\underline{X2}'\underline{X2}^+}(\zeta,t-\underline{\tau}_{rA2})R_{\underline{X3\dot{X}3}^-}(t-\underline{\tau}_{rA2},\zeta)$$

$$- \dot{E}[X1^{+}(\theta)]R_{\underline{X2\dot{X}2}^{-}}(t-\underline{\tau}_{rA2},\zeta)R_{\underline{X3}'\underline{\dot{X}3}^{+}}(\zeta,t-\underline{\tau}_{rA2})$$

$$- \dot{E}[\underline{X1}^{+}(\theta)]R_{\underline{\dot{X}2}^{+}\underline{\dot{X}2}^{-}}(t-\underline{\tau}_{rA2},\zeta)R_{\underline{X3X3}'}(t-\underline{\tau}_{rA2},\zeta)$$

$$+E[\underline{X1}(\theta)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})R_{\underline{\dot{X}3}^{+}\underline{\dot{X}3}^{+}}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})$$

$$+ E[\underline{X1}(\theta)R_{\underline{X2}'\underline{\dot{X}2}^{+}}(\theta-\underline{\tau}_{r01},t-\underline{\tau}_{rA2})R_{\underline{X3\dot{X}3}^{+}}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})$$

$$+ E[\underline{X1}(\theta)]R_{\underline{X2\dot{X}2}^{+}}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})R_{\underline{X3}'\underline{\dot{X}3}^{+}}(\theta-\underline{\tau}_{r01},t-\underline{\tau}_{rA2})$$

$$+ E[\underline{X1}(\theta)R_{\underline{\dot{X}2}^{+}\underline{\dot{X}2}^{+}}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})R_{\underline{X3X3}'}(t-\underline{\tau}_{rA2},\theta-\underline{\tau}_{r01})\}d\zeta d\eta d\theta$$

$$- \int_0^{t-\underline{\tau}_{fA2}}\int_0^{\theta-\underline{\tau}_{rA2}}\int_0^{\theta-\underline{\tau}_{f01}} \{\dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2X2}}(0,t-\underline{\tau}_{rA2})\dot{E}[\underline{X3}^{+}(t-\underline{\tau}_{rA2})]$$

$$+ \dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2\dot{X}2}^{+}}(0,t-\underline{\tau}_{rA2})E[\underline{X3}(t-\underline{\tau}_{rA2})]+\dot{E}[\underline{X1}^{-}(\Theta)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},0)R_{\underline{X3\dot{X}3}^{+}}(0,t-\underline{\tau}_{rA2})$$

$$+ \dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2}'\underline{\dot{X}2}^{+}}(0,t-\underline{\tau}_{rA2})R_{\underline{X3X3}}(0,t0\underline{\tau}_{rA2})$$

$$\dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2X2}'}(t-\underline{\tau}_{rA2},\eta)R_{\underline{\dot{X}3}^{+}\underline{\dot{X}3}^{+}}(t-\underline{\tau}_{rA2},\eta)$$

$$+ \dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2}'\underline{\dot{X}2}^{+}}(\eta,t-\underline{\tau}_{rA2})R_{\underline{X3\dot{X}3}^{+}}(t-\underline{\tau}_{rA2},\eta)$$

$$+ \dot{E}[\underline{X1}^{-}(\theta)]R_{\underline{X2\dot{X}2}^{+}}(t-\underline{\tau}_{rA2},\eta)R_{\underline{X3}'\underline{\dot{X}3}^{+}}(\eta,t-\underline{\tau}_{rA2})$$

$$+ \dot{E}[\underline{X1}^-(\theta)]R_{\underline{\dot{X2}}^+\underline{\dot{X2}}^+}(t-\tau_{rA2},\eta)R_{\underline{X3X3}'}(t-\tau_{rA2},\eta)$$

$$- \dot{E}[\underline{X1}^-(\theta)]R_{\underline{X2X2}'}(t-\tau_{rA2},\zeta)R_{\underline{\dot{X3}}^+\underline{\dot{X3}}^-}(t-\tau_{rA2},\zeta)$$

$$- \dot{E}[\underline{X1}^-(\theta)]R_{\underline{X2}'\underline{\dot{X2}}^+}(\zeta,t-\tau_{rA2})R_{\underline{X3\dot{X3}}^-}(t-\tau_{rA2},\zeta)$$

$$- \dot{E}[\underline{X1}^-(\theta)]R_{\underline{X2\dot{X2}}^-}(t-\tau_{rA2},\zeta)R_{\underline{X3}'\underline{\dot{X3}}^+}(\zeta,t-\tau_{rA2})$$

$$- \dot{E}[\underline{X1}^-(\theta)R_{\underline{\dot{X2}}^+\underline{\dot{X2}}^-}(t-\tau_{rA2},\zeta)R_{\underline{X3X3}'}(t-\tau_{rA2},\zeta)$$

$$+ E[\underline{X1}(\theta)]R_{\underline{X2X2}}(t-\tau_{rA2},\theta-\tau_{f01})R_{\underline{\dot{X3}}^+\underline{X3}^-}(t-\tau_{rA2},\theta-\tau_{f01})$$

$$+ E[\underline{X1}(\theta)]R_{\underline{X2}'\underline{\dot{X2}}^+}(\theta-\tau_{f01},t-\tau_{rA2})R_{\underline{X3\dot{X3}}^-}(t-\tau_{rA2},\theta-\tau_{f01})$$

$$+ E[\underline{X1}(\theta)]R_{\underline{X2\dot{X2}}^-}(t-\tau_{rA2},\theta-\tau_{f01})R_{\underline{X3}'\underline{\dot{X3}}^+}(\theta-\tau_{f01},t-\tau_{rA2})$$

$$+ E[\underline{X1}(\theta)]R_{\underline{\dot{X2}}^+\underline{\dot{X2}}^-}(t-\tau_{rA2},\theta-\tau_{f01})R_{\underline{X3X3}'}(t-\tau_{rA2},\theta-\tau_{f01})\}\, d\zeta d\eta d\theta.$$

The above expression contains 75 terms. As pointed out in Sec. 3.3, the various correlation functions can be determined from a much smaller set of basic correlation functions.

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